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# Mathematical Modelling of Mid-term Options Price of Ijārah Sukūk

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A thesis presented for the degree of  
Doctor of Philosophy

Department of Mathematics

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# Declaration

I hereby declare that this thesis has not been and will not be, submitted in whole or in part to another University for the award of any other degree.

Signature:

# Dedication

Dedicated to my Late Father

Dayfallah bin Aziz bin Hendi Alsolami

who said

*“Walking in life without knowledge is like walking in the  
woods without a weapon.”*

# Abstract

The main aim of this thesis is to study the pricing of options of Ijārah Sukūk for lifespan. The pricing formulae of mid-term call and put options are derived by computing the expected value under the risk neutral measure and using an appropriate condition of exercising the option at mid-term. The mid-term option prices with continuous Ijārah obtained using these formulae are compared with the prices of European and American options with dividend for lifespan. The comparison is done both analytically and numerically. The same analysis is done for callable and puttable Sukūk with Ijārah and compared with the prices of European and American callable and puttable bond with coupon for lifespan. We also study the relationship between callable Sukūk price and Ijārah rate by computing the duration and convexity of the callable Sukūk price. The same analysis is done for puttable Sukūk.

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# Chapter 1

## Introduction

### 1.1 An Overview of Islamic Finance

Islamic finance is an alternative approach to conventional finance [6]. This kind of finance is based on Shariah which is derived from the Qur'an(holy book) and Hadith (sayings of prophet Muhammad). Therefore, major principles of Shariah Law that are applicable to finance and that differ from conventional finance are [6, 19]:

- (a) **Prohibition on interest (riba):** In Islamic finance, paying or receiving interest on loan is prohibited. It bans any kind of investment in debt. The lender can still make profit in a sharia compliant way. For example, instead of giving loan for a property, a bank buys the property and leases it to the investor and receives rent as opposed to the interest from the investor.
- (b) **Prohibition on uncertainty:** There is no scope of any uncertainty in the terms and conditions of a contract. In other words, it is necessary that all of the terms and conditions of the risks are clearly understood by all parties. This helps in eliminating all speculative transactions which involve gharar (excessive uncertainty).

- (c) **Risk and profit/loss sharing:** It is compulsory in Islamic finance that all parties involved in a financial transaction share both the associated risks and profits.
- (d) **Ethical investments that enhance society:** The Islamic finance is not only investment but also a way to enhance the community to growing up and meet their needs. Furthermore, the Qur'an prohibits the use of alcohol, pornography, gambling, and pork. Any kind of Investment in such industries are discouraged.
- (e) **Tangible underlying asset:** There is always some tangible underlying asset associated with each financial transaction, which means that all the transactions involve real economic activities. This ensures that the risk is borne by the creditors and that they cannot transfer the risk to someone else.

Hence, Islamic finance provides alternative methods for capital formation and economic development. Therefore, Islamic finance is resilient to shocks because of its emphasis on risk sharing, limits on excessive risk taking, and strong link to real activities [24].

## 1.2 Introduction

Sukūk are securities of equal denomination representing individual ownership interests in a portfolio of eligible existing or future assets that are Shariah-compliant [30]. Ijārah Sukūk are the securities signifying ownership of well-defined existing and known assets linked with a lease contract, the rents of which is payable to Sukūk holders [26]. Musharakah Sukūk is a partnership between entities, where each entity has to contribute to the partnership in the form of either cash or in kind [56]. A proper understanding of the Sukūk and its various types is of huge interest to policy makers and banks working in the realm of Islamic securities and Shariah Law. There are many instances where efforts in this domain have helped poor people in some countries to gain financial independence

and leave poverty. For example, Nobel Peace Prize was conferred on Professor Muhammad Yunus and the Grameen Bank for their pursuits in the field of microfinance and Islamic securities which eventually helped the economically deprived section of people in Bangladesh. The financial tools used in these process are microfinance by Ijārah Sukūk and Musharakah Sukūk, yet most people working in finance in the world have limited insight [22].

The Middle East, North Africa, and Southeast Asia were the hubs where modern systems in Islamic financing and securities were developed. The first tenet of Islamic financing is the prohibition of *riba* (interest). Furthermore, the transactions must have an underlying tangible asset. Among the various Islamic securities techniques, the product based Sukūk are the most popular, and in recent years, it has been accepted as a viable alternative. Therefore, Sukūk are Islamic bonds that are structured to generate returns to investors without breaking the Islamic law [7].

There are many motivations that encourage me to choose this kind of subject in my thesis. The subject of Sukūk is very important for financial markets as this new kind of bond is growing rapidly in the world market [37]. To help further in this growth we need more investigations and studies for this kind of bond by the researchers in economics and mathematics and which must also incorporate the interpretations by the Shariah scholars to economic or mathematical models. In addition, the Shariah scholars have a lot of new ideas to improve the Sukūk structures to be agreed with Islamic law. These ideas can further be taken up by economic and mathematical researchers in this area. This is one of the reasons which motivated me to study the main idea in this thesis.

Similar to European or American call and put options on an underlying asset, we can consider options of Sukūk or Sukūk options. In this thesis we consider mid-term options of Ijārah Sukūk that can be exercised in the middle of the term, as well as the end of the term of the contract if not exercised at mid-term. The pricing formula of

mid-term call and put option is derived by computing the expected value under the risk neutral measure and using an appropriate condition of exercising the option at mid-term. A European option can be exercised only at maturity but an American option gives its holder the right to exercise at any time up to and including maturity. The holder of a mid-term Sukūk option has a right to exercise at mid-term or at maturity. Therefore, the price of the mid-term Sukūk option depends on the condition whether to exercise at mid-term or not. This condition is interpreted in mathematical terms, and the price of mid-term Sukūk option is obtained by computing the risk neutral expectation of the payoff conditional to this restriction. Furthermore, the mid-term option prices with continuous Ijārah obtained using this formula are compared, both analytically and numerically, with the prices obtained for European and American options with continuous dividend and for the lifespan of the option. The analysis in the case of mid-term call option of Sukūk is done in Chapter 3, while Chapter 4 deals with the mid-term put option of Sukūk.

In Chapter 5, I discuss callable and puttable Ijārah Sukūk and obtain their prices using the corresponding prices of mid-term call option and put option of Sukūk. Furthermore, I compare the prices of callable and puttable Sukūk with the prices of American and European callable and puttable bonds. I also study the duration and convexity of the callable and puttable Sukūk to see the sensitivity of price to change in Ijārah rate. In Chapter 6, I summarise the results obtained in this thesis.

# Chapter 2

## Literature Review

### 2.1 Literature Review

Several studies have been conducted globally by researchers on Sukūk, the research efforts focus on various themes and parameters involved in the domain of Islamic financial products. The studies have analyzed the key tenets of Shariah Law, basic concepts regarding Islamic finance, legal aspects, business scenarios, government policies, involvement of central banks. Furthermore, they studied the relevant parameters to aid the development of Islamic banking products, securities prices as Sukūk prices and Sukūk options prices.

#### 2.1.1 Evolution of Studies on Sukūk

The studies conducted on Sukūk have evolved over a period of time in terms of their focused areas and applicability to the realm of banking. The pattern of studies through the early 1990's is completely different to recent research initiatives. A brief analysis of the themes of study during the various periods has been presented below:

The initial studies in this field have been generally dedicated on the understanding and elucidation of the basic theory of Islamic securitization and the association with the idea of Sukūk [3, 43]. Tahir (1994) studied a new approach to product development in Islamic finance which is the Islamic securitization (Sukūk). He proposed a product development template to develop more customer-need centric financial products in Islamic financial world. This template was a step-by-step process of developing Islamic financial products and is derived by following the best practices of product development in the engineering domain. This methodology aims to formalize and improve the product development process in Islamic finance. As an illustration of the proposed template, Ijārah Sukūk is developed for Islamic finance [50]. The other studies in literature revolved around the procedures, stakeholders involved, and the perceptions and viewpoints of various scholars. The studies also looked at innovation and explored the various financial instruments and products [58, 4]. Further research was also done on new product development with respect to Shariah Law and the procedures to modify conventional products to be compliant with Shariah Law. Studies have also identified that the support of government policy gives businesses freedom in developing new products and services [43].

Later research efforts were based on legal aspects and issues with respect to Shariah Law [3, 55]. Even though concerns associated with Islamic law were deliberated earlier. Many specialized research has been carried out since the remarkable growth in the market of Sukūk, with different new products created and unique contracts being utilized [38]. Generally, the studies have explored several issues like the issuance of Sukūk, relationship between the stakeholders, applicability of contracts based on Shariah Law, and the identity and positions of the stakeholders involved [59, 55].

The studies done recently explore the possibilities involved in standardization, harmonization and globalization of Sukūk [1, 3, 60]. The studies provide views on creation of standardized Sukūk by explaining the views of Shariah on several standard issues.

The issues are generally linked to the procedure of issuance and the nature of the underlying assets [60]. The studies also examined the importance of the Central Banks, regulatory bodies, and auditing/advisory board of Islamic Financial institutions [21, 36]. Furthermore, some studies explore how the Sukūk could exploit the potentials within such an increasingly competitive International market. Also, studies try to look at the , challenges, risks and limitations of the Sukūk, where there is a huge gap between Sukūk demand and supply due to increasing demand of Sukūk [1].

### **2.1.2 Growth of Sukūk**

The Islamic Financial Market has continued to grow steadily and has even out-paced conventional products in spite of a global slowdown [47]. The worldwide demand for Islamic financial services grew by 12% in 2014 to reach a record high and become a \$2 trillion market. Since the economic crisis of 2007-08 the market size has tripled. In the last three years the issuances of Sukūk has crossed the USD 100 billion mark, and it now boasts of sovereign issuers from Muslim as well as non-Muslim jurisdictions [2]. We see also the Islamic finance and Sukūk grows in non-Muslim jurisdictions because it does not violate any local regulations as it only deals with real economic transactions. The tremendous growth is of great significance as Sukūk and Sukūk funds are looked at as a resilient alternative to conventional products [33].

The demand for Sukūk has also been stimulated by the financing needs of sovereigns, multilateral agencies and corporates. Currently, Sukūk provides a technique to investors and issuers to control liquidity, offer asset and infrastructure financing, and an option to diversify the investment portfolio to offset the volatility of other assets [33].

The government of United Kingdom took the lead in the issuing of the Ijara Sukūk to the amount of GBP 200 million. This was then followed by various governments around the world, like Luxembourg, Senegal, Hong Kong and South Africa, each

amounting to EUR 200 million, XOF 100 billion, USD 1 billion and USD 500 million, respectively. Currently, the Sukūk are mostly issued by the central government and central banks [2]. However, the growth trends and increasing awareness suggests that the market is ripe for a strong entry by the government sector [33]. In Europe, the recent tightening of liquidity in the conventional markets has caused a spur of interest in other forms of funding, and the Sukūk might have a role to play in the European markets as well [61].

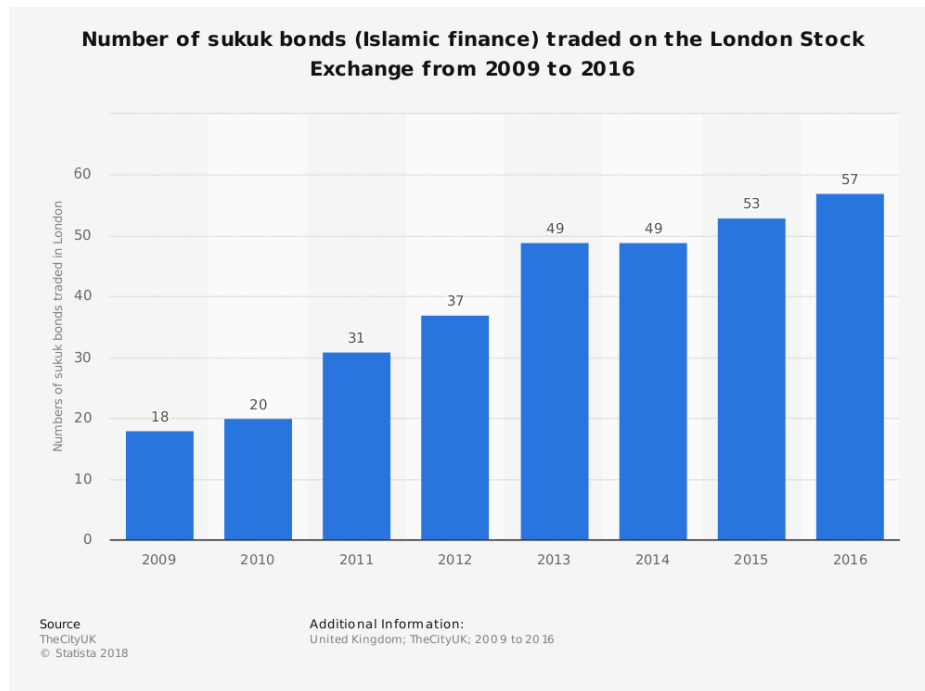


Figure 2.1: Sukuk traded on the London stock exchange [61]

In the short term future, it is likely that Sukūk might convert into a key constituent of the financial world in both the developed, as well as the developing economies. One of the major factors contributing to this progress is the ever-growing necessity for financing of infrastructure projects globally, and sovereign requirements in terms of liquidity. Sukūk products are nicely positioned to play a pivotal role in meeting the funding gap [33]. However, in the last two years during the falling of oil price, the Sukūk market in most of oil countries is rising rapidly because these countries want to cover the budget deficit and complete the infrastructure projects as in Saudi Arabia and Kuwait [5].



Nafith AL-Hersh (2014) studied Islamic finance in global markets depending on global Islamic finance assets , global Islamic funds sector and global Islamic Sukūk sector. He expected that Islamic banking assets will grow at a compounded average growth rate of 20% over (2013–2020) across the countries to reach USD 3.034 trillion by 2020 (see Figures 2.2 and 2.3. Note that Sukūk is the most interesting investment instrument in Islamic finance [39].

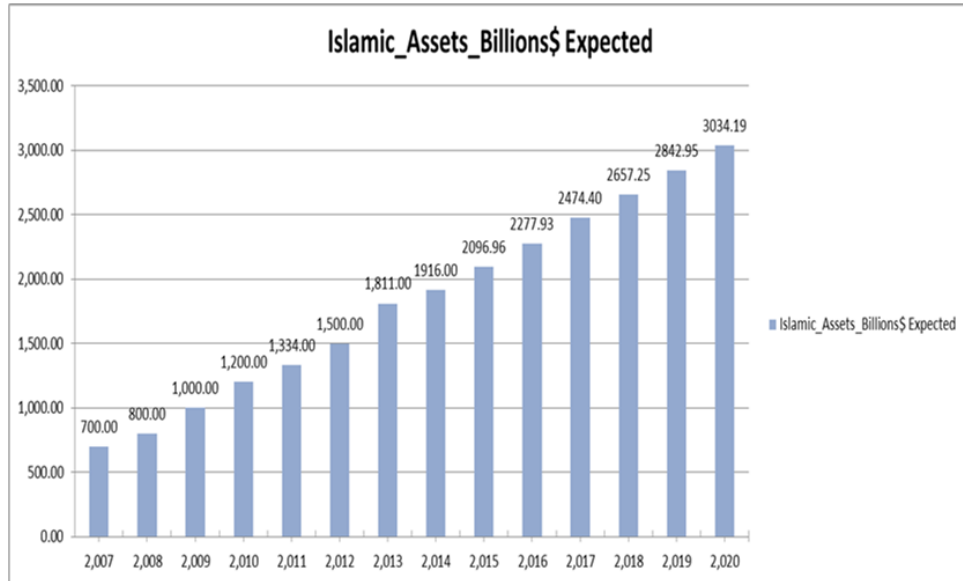


Figure 2.2: Expected Islamic assets growth in 2020 [39]



Source: Zawya, IFIS, Bloomberg, KFHR

Figure 2.3: Global Islamic assets growth trend [39]

### 2.1.3 Standardisation of Sukūk

One of the major requirements for the growth of Sukūk is the standardisation with respect to pricing controls and rating of Sukūk funds. The developments in this regard have assisted in developing a global reach for Sukūk and Islamic financial products [41].

The London Interbank Offered Rate (LIBOR) was the general benchmark adopted in determining the price of Sukūk profit or Ijārah-based return, which is undesirable under the Shariah law because it is an interest-based benchmark [42]. As such, the Sukūk pricing method implements a similar pricing instrument as bonds, using the variance between bid and ask prices (or bid-ask spread), and the volume of trading. Sukūk returns result due to lease, profit or sales of assets such as property/business. The currently used technique of separating the pricing procedure for Sukūk from conventional financing benchmarks in the debt market is a major challenge in the Islamic capital market. Thus, it is tedious to create a base for pricing Sukūk on a regular basis to enable trading of Sukūk [41].

In 2002, the Islamic International Rating Agency was established. It was possible due to the coordination between the Bahrain Monetary Agency and the Islamic Development Bank and other stakeholders involved [16]. This enabled the presence of an agency to independently evaluate, analyze and rate Islamic banks and products. So, an initial effort in establishing a standard for implementation, approval and innovation in Islamic financial products was taken in the developing Sukūk capital markets. This was followed by the entry of Moody Investors Service which is the one of the biggest three credit rating agencies in the world, who developed a unique rating methodology for Sukūk. This rating tool analyzed the association of Sukūk with the income gained from the underlying assets. It was determined that the income generated could be based on an asset if the payment was initiated through a contract for repurchase, even though under

Islamic financial instruments all transactions mostly incorporate a group of underlying risk on assets which is divided amongst the issuer and the Sukūk holders [3].

The one possibility to estimate the risk-free Ijārah rate is the inter-bank lending rate. This appears to be premised on the basis that these banks benefit from an implicit guarantee, underpinned by the role of the monetary authorities. Therefore, in 2011, some Islamic banks from Saudi Arabia, United Arab Emirates, Bahrain, Qatar, Kuwait and Pakistan in conjunction with Thomson Reuters established an Islamic interbank benchmark rate, IIBR, in US dollars [46]. The IIBR represents an average expected cost of short-term interbank market funding for the Islamic banking industry that can be used as a base in the pricing of interbank market transactions, retail and corporate Islamic financial products, and Sukūk and other fixed income structures . The benefit of having this Sharia-compliant risk-free interest rate (risk-free Ijārah rate as in Ijārah Sukūk) instrument or denominated in US dollars is to encourage consistency across all contributors, let the Islamic financial products more globally integrated, and promoting Islamic capital markets. Furthermore, with the recent growth of the Sukūk and the heightened interest in the Islamic financial markets and products worldwide, the use of the IIBR is expected to gradually increase [42].

#### **2.1.4 Sukūk - A European Perspective**

In recent times there has been continued efforts in the European market to encourage Islamic finance, predominantly in the United Kingdom, Ireland and Luxembourg. This has assisted the development of the Islamic finance and especially issuances of Sukūk in geographies apart from Malaysia and the Middle East [33].

Among the European economies, the United Kingdom is the centre for Islamic finance, it stems from the already eminent role of London in global finance and a necessity to meet the requirements of its own Muslim population. In a report released by Thom-

son Reuters focussing on Islamic finance, the United Kingdom was ranked as the highest non-majority Muslim country in terms of education and research, governance structure, corporate involvement in social issues, and general awareness [25]. The sovereign Sukūk issued by the United Kingdom government was worth GBP 200 million, with an oversubscription greater than eleven times and orders were placed to the tune of GBP 2.3 billion from a variety of geographies and financial institutions. The sovereign Sukūk issued by the United Kingdom government opened a source for the infusion of funds, but it also enabled the domestic banks involved in Islamic finance to utilize it to meet liquidity and capital requirements. The issuance was also an avid demonstration that a proper structure exists in the United Kingdom to issue Sukūk on similar lines to conventional bonds [29].

The issuance of sovereign Sukūk in the United Kingdom was a trigger for further developments in the field of Islamic finance [29]. One development was the announcement by the United Kingdom Export Finance to provide guarantees to British exporters for Islamic finance products. In 2015, the United Kingdom Export Finance declared that it would guarantee Sukūk issued by Emirates (Dubai airline), this was a major milestone and had a noticeable impact on “buyer credits” with Export Control Agency guarantees which facilitated exporters to provide medium term funding to markets as constituent of their project bids within Organization of Islamic Conference (OIC) countries [33].

Another development occurred in 2014, when Sukūk was issued by the International Finance Facility for Immunization, for the Vaccine Alliance, and they raised about USD 500 million. It was one of the foremost Sukūk dedicated to social issues which facilitated the vaccination of children to protect them from preventable diseases. Its listing on the London Stock Exchange led to an oversubscription with involvement of large number investors from a global base [29].

The resolve of the United Kingdom government to convert into a major force

in countries of the Islamic world was possible only due to the conducive environment and architecture. In October 2013, the then British Prime Minister David Cameron in his keynote address at the ninth World Islamic Economic Forum held in London said “Already London is the biggest center for Islamic finance outside the Islamic world, and today the ambition is to go further still. Because I don’t just want London to be a great capital of Islamic finance in the Western world, I want London to stand alongside Dubai and Kuala Lumpur as one of the great capitals of Islamic finance anywhere in the world” [14]. The majority of the leading law firms and professional service firms in London provide services covering all facets of Islamic finance and this provides the capability to structure Islamic financial products for the global market from its base in London. The Asian and Middle Eastern firms working towards expanding their base of investors and their aim to facilitate activity in the secondary markets of Sukūk are heading towards Europe. Also, the distribution of several Islamic funds has originated from London and areas where experienced local and global firms are present to enable participation of investors globally [33].

The future of the Sukūk market in Europe depends on the size of new and global issuance of Sukūk by sovereign, supra-national and private players. The growth of Islamic finance in Europe and the strong support structure in place bodes well for further listings and growth of Sukūk funds [29]. Furthermore, in Europe, the requirement for alternate sources of funding is ever-growing due to the need for infrastructure development and Sukūk funds are nicely placed to play a major role in this scenario. Presently, the infrastructure projects relevant to Islamic investment are growing on the domestic front but with the active involvement of the United Kingdom government it is possible that it may provide a strong base for development and project funding across Ireland, Wales, Scotland, and other areas in Europe [29, 33].

### 2.1.5 Comparison of Sukūk with Conventional Bonds

There exist certain similarities between Sukūk and conventional bonds, however, significant differences also exist. A comparison based on an analysis of various parameters has been explained below[27]:

	<b>Conventional Bonds</b>	<b>Sukūk</b>
Asset ownership	Debt obligation from the issuer to the bond holder.	The investors are given the ownership in the assets.
Investment criteria	Can be used to finance any asset/project that complies with the domestic regulations.	The assets on which the Sukūk are based should be following the Shariah Law.
Issue unit	Each conventional bond represents a share of the debt.	Each Sukūk represents a share of an underlying asset.
Issue price	The face value is dependent on credit rating or worthiness of the issuer.	The face value is based on the market determined value of the underlying asset.
Investment rewards and returns	Generally in the form of regular payments of interest for the duration of the bond and the principal amount is returned on maturity.	The holders receive a share of profits from the underlying assets or Ijārah of assets as in Ijārah.
Effects of costs	The holders are not impacted by costs of the asset/project it supports and the performance of underlying assets does not affect returns of the investor.	In most kind of Sukūk, the holders are affected by costs with respect to the underlying asset and these have an effect on the profits, but Ijārah Sukūk are not affected because the Ijārah is fixed from the beginning of the contract and there is no transaction cost.

Table 2.1: Comparison of Sukūk with Conventional Bonds.

### 2.1.6 Differences between Ijārah (Rent) and Dividend

Although Ijārah (Rent) can be considered as playing a similar role for Sukūk option as dividend in case of European options, but there are some differences between these two kinds of payments. In Table 2.2, we summarize the main points of differences

between these two concepts:

<b>Ijārah</b>	<b>Dividend</b>
It is always agreed in advance.	It changes as per changes in profit.
Non-payment of Ijārah (rent) leads to default.	Dividend is not paid in case of loss.
The Ijārah remains same even if the business becomes better.	The dividend can increase if the business gets better.
There is a negative impact of inflation.	Inflation should bring long term positive impact.

Table 2.2: Comparison between Ijārah and dividend.

### 2.1.7 Volatility of Sukūk

Sukūk, like any market-linked financial product, is systemically vulnerable to volatility in a multitude of factors. Several studies have been carried out to analyze the impact of various parameters on volatility, measures to mitigate risks and steps required to build an efficient portfolio of Islamic financial products[52].

A study shows that there exists a high degree of correlations between Sukūk and stock markets, especially the domestic markets, and the markets of United States and Europe. Scholars also point out that Sukūk can be included as part of a diversified portfolio as research suggests that they are less volatile in equities [8, 40].

Some studies have focused on comparing the market performance of Sukūk with respect to other conventional products. Their results show that conventional bonds are riskier than Sukūk but the government bonds are better than Sukūk on the basis of volatility. Also, certain studies show that Sukūk are instruments that lie midway between bonds and equity in terms of volatility. Furthermore, like conventional stocks, Sukūk volatility were also found to be linked with market liquidity factors and prices of crude oil [9, 45].

### 2.1.8 Options in Sukūk

In options, the investor gains the right to buy or to sell a pre-defined quantity of an asset at a specific price on a definite date in the future, however the investor is not obligated to carry out the transaction. Generally, there are two types of options: call and put, the former gives the investor to sell while the latter gives the right to buy [56].

In Islamic financial products, the option contracts utilize the theory of Wa'ad (the right but not the obligation). Therefore, in Sukūk (Islamic bond), option is an option to buy or sell assets at an agreed price at a pre-defined price on a certain date when the contract matures, or there is a default. In a Sukūk, as there is always a tangible asset on which transactions take place, there exists a susceptibility to a variations in the asset value. Therefore, the utilization options enables a reduction in the risk involved [56].

With regard to option's exercise, the Sukūk option style currently can only be exercised at the option's expiration date which is similar to European exercise option style. However, exercise option at mid-term is possible in Ijārah Sukūk and this is what we want to derive and discuss in the next chapters [35, 53].

## 2.2 Ijārah Sukūk

The Ijārah Sukūk is the most frequently used Sukūk structure also called as Sukūk al-Ijārah. The Ijārah Sukūk boasts a world-wide approval. Several scholars have defined it as the classical structure of Sukūk based on which other similar structures originated [26]. The highly simple structure of the Ijārah Sukūk and its acceptance amongst Shariah scholars is also an important reason for its popularity. In the domain of Islamic financial instruments, the term Ijārah is generally defined as the 'allocation of the usufruct of the underlying tangible asset to another individual in exchange for a Ijārah



requested from tenant, in a more literal sense it is a form of lease. The Ijārah is generally utilized in a way such that it enables payments regularly during the entire duration of the Sukūk. Furthermore, it offers inherent flexibility to modify the structure of payments to generate a profit. Also, an undertaking for repurchase is generally approved for Ijārah Sukūk without objections from Shariah Law. Thus, it makes it simple to implement Ijārah structure as the cornerstone for issuance of Sukūk [56].

In this thesis, I will concentrate only in Ijārah Sukūk. Ijārah Sukūk are popular financial instruments in Islamic financial system. Also, Ijārah Sukūk is the first kind of Islamic securities that was developed early and its issuance is rapidly becoming the most popular [26]. Therefore, there are many motivations that encourage individuals and corporations to use this kind of Sukūk. One important point of these motivations is that the return from the Sukūk to the investors is known and remains constant which comes from the Ijārah payments, while the return from other Sukūk like Musharakah Sukūk is unknown because return is based on the profit or loss in the Sukūk projects. Therefore, this thesis will deal with the mid-term options price with Ijārah Sukūk only.

### **2.2.1 Structure of Ijārah Sukūk**

We now explain the structure of an Ijārah Sukūk (see Figure 2.4). Usually, the Sukūk are used to finance real estate projects, private equity and to finance machinery or equipment. For this purpose, first the issuer of Sukūk needing funds (e.g. Project Company) establish a special-purpose-vehicle SPV (issuer / funding company) and transfers assets to this SPV which holds the legal rights to these underlying assets. These underlying assets must be clearly identified. In some Ijārah Sukūk, the underlying asset owner leases the underlying assets to the issuer and rents the underlying asset back, thereby avoiding a true sale of underlying assets. The SPV issues Sukūk certificates to investors, and investors give cash to the issuer in return as investments. The project company

leases the assets back from the SPV under a lease agreement and pays rents (Ijārah) to the SPV for the leased assets. The amount of each rental is equal to the periodic distribution amount payable under the Sukūk at that time. This amount may be calculated by reference to a fixed rate or variable rate. On account of any default, the project company undertakes to purchase the assets from the SPV at the outstanding principal amount [50].

The Ijārah and period of Ijārah must be agreed in the beginning of contract. Since ownership of the underlying remains with the SPV, the liabilities arising from the ownership also rest with the SPV. But any liabilities relating to the use of the underlying lies with the project company. Moreover, the project company cannot use the underlying for any purpose other than the purpose mentioned in the Ijārah contract. Ijārah must be determined at the time of contract for the whole period of the contract. However, it is possible to split the term of the Ijārah into smaller rental periods where each rental period is considered as a separate lease [56].

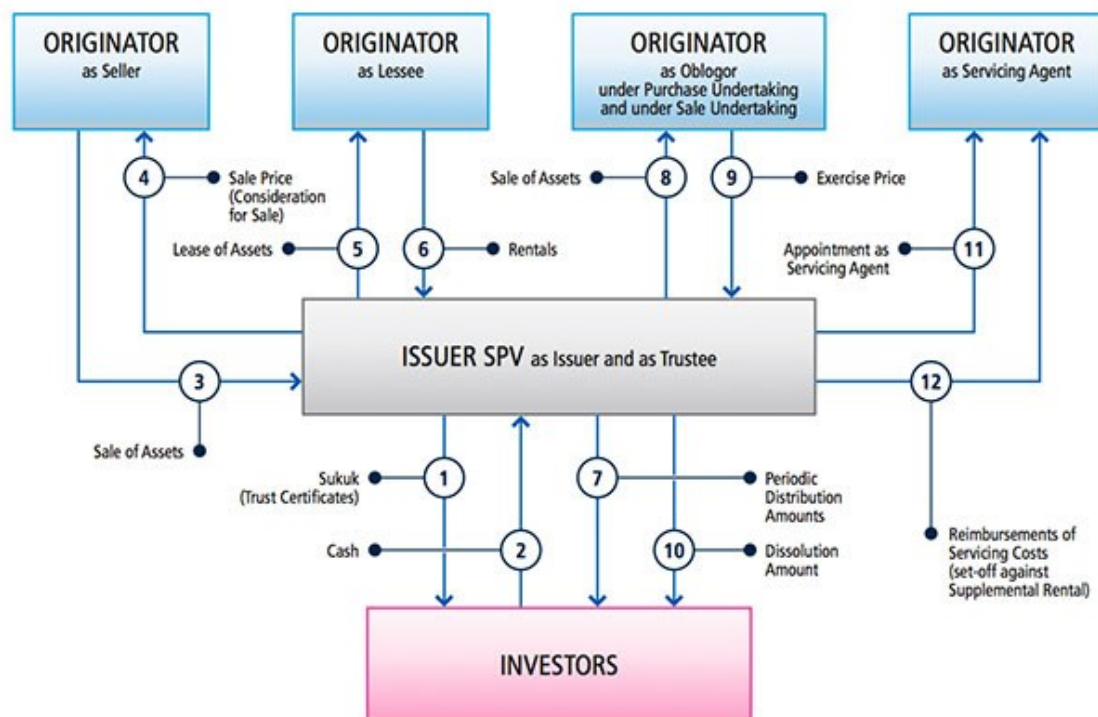


Figure 2.4: Structure of Ijārah Sukūk [56].

## **2.3 Overview of Other Types of Sukūk than the Ijārah Sukūk**

### **2.3.1 Musharakah Sukūk**

The term Musharakah is derived from the Arabic word “shirkah”, which means partnership. Thus a basic definition of Musharakah is a partnership between entities, where each entity has to contribute to the partnership in the form of either cash or in kind [26]. So it is similar to a venture between two entities but through arrangements it can also gain legal validity. Finally, the entities involved in the partnership divide the returns in quantities as decided earlier and take the losses of the partnership in proportion to their investment of capital. In the realm of Islamic financial products, generally two types of structures are followed for the Musharakah arrangements: Shirkat al-’Aqd and Shirkat al-Melk, the former is usually referred to as the ‘business plan’ Musharakah, and the latter is usually referred to as a ‘co-ownership’ [56].

### **2.3.2 Murabahah Sukūk**

The term Murabahah is generally used to refer to a contractual agreement between an issuer and an investor wherein the issuer would sell definite assets to the investor for immediate delivery while having the expectation that the investor will oblige with the payment schedule under the arrangement [26]. The calculation of the price takes into account the cost price at which the issuer purchased the assets, and a previously decided margin demonstrating the returns gained from in the transaction. The schedule of payments can be decided to be periodic as agreed upon initially. So, it assists in building up stream of regular income for the issuer during the entire term of the contract [56].

### 2.3.3 Istisna'a Sukūk

The term Istisna'a refers to ordering a manufacturer to produce a certain object for the customer. In such an agreement, it is imperative that the price and the characteristics of the object to be produced are well defined and mutually agreed on at the onset [26]. Presently, the Istisna has grown into an instrument to fund the construction involved in projects. Generally, in an Istisna'a Sukūk, an Istisna agreement is used in conjunction with future looking lease agreement. However, it should be analyzed that there exists criticisms to certain payment approaches being used under this Sukūk. Furthermore, there exists no Istisna'a Sukūk listed on NASDAQ and Dubai, and it has not developed into an alternative source for Islamic funding for infrastructure projects [56].

# Chapter 3

## Mathematical Modelling of Mid-term Call Option Price of Ijārah Sukūk

### 3.1 Overview

The purpose of issuing Sukūk is generally to finance large and long-term projects, such as construction of massive infrastructure and large industrial production facilities. Therefore, Sukūk contracts usually have long lifespans of 10 to 30 years. Hence, most buyers and sellers of options of Sukūk seek options to exercise before maturity. Option derivatives of Sukūk market currently is exercised only at expiry as the European option [18]. However, mid-term exercising of the call option is possible in Ijārah Sukūk [35, 53]. We introduce the mathematical model of a mid-term exercise option in this chapter to demonstrate the pricing possibilities of this idea.

**Definition 3.1.1.** *Sukūk* are securities of equal denomination representing individual ownership interests in a portfolio of eligible existing or future assets that are Shariah-

compliant [30].

**Definition 3.1.2.** An *Ijārah* is an agreement made by an institution offering Islamic financial services (IIFS) to lease to a customer an asset specified by the customer for an agreed period against specified installments of lease *Ijārah* [26]. An *Ijārah* contract commences with a promise to lease that is binding on the part of the potential lessee prior to entering the *Ijārah* contract.

**Definition 3.1.3.** *Ijārah Sukūk* are the securities representing ownership of well-defined existing and known assets tied up to a lease contract, *Ijārah* of which is the return payable to *Sukūk* holders [26].

We establish a model along the concept of *Ijārah Sukūk*, each of them representing ownership of well-defined existing and known assets tied up to a lease contract, *Ijārah* of which is the return payable to *Sukūk* holders. On the other hand, buyer of *Sukūk* has a right to buy the underlying asset at mid-term time  $T/2$  and at the expiry time  $T$  if not exercised at mid-term.

## 3.2 Mathematical Background

In this section, we give some definitions concerning filtration and Itô's formula. We also explain how geometric Brownian motion is used in the simulation of underlying asset prices. Finally, we study the basic aspects of stochastic differential equations and relevant properties that we will use in our mathematical modelling of call and put option of *Ijārah Sukūk*.

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space.

**Definition 3.2.1.** Let  $(E, \mathcal{E})$  be a measurable space. A function  $f : \Omega \rightarrow E$  is called  $\mathcal{E}$  -

measurable if for every  $B \in \mathcal{E}$ ,

$$f^{-1}(B) := \{\omega \in \Omega : f(\omega) \in B\} \in \mathcal{F}. \quad (3.1)$$

A random variable  $X : \Omega \rightarrow \mathbb{R}$  is a  $\mathcal{B}(\mathbb{R})$ -measurable function where  $\mathcal{B}(\mathbb{R})$  is the Borel  $\sigma$ -algebra on  $\mathbb{R}$  [49].

**Definition 3.2.2.** For any bounded random variable  $X$  the *expected value (mean)* of  $X$  on  $(\Omega, \mathcal{F}, \mathbb{P})$  is the integral of  $X$  with respect to the measure  $\mathbb{P}$  [12],

$$\mathbb{E}[X] = \int_{\Omega} X d\mathbb{P} = \int_{\Omega} X(\omega) \mathbb{P}(d\omega). \quad (3.2)$$

**Definition 3.2.3.** A random variable  $X$  is *continuous* if there exists a density function  $f(x)$  such that [12]

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f(x) dx. \quad (3.3)$$

**Definition 3.2.4.** The *variance* of a random variable  $X$  is

$$\text{Var}(X) = \mathbb{E}[(X - \mu)^2], \quad (3.4)$$

where  $\mu = \mathbb{E}[X]$  is the mean of  $X$  [12].

**Definition 3.2.5.** If  $X$  a standard normally distributed random variable, then the random variable  $Y = e^X$  follows a log-normal distribution with density [12]

$$f_Y(x) = \frac{1}{x\sqrt{2\pi}} e^{-\frac{(\log(x))^2}{2}}. \quad (3.5)$$

**Definition 3.2.6.** A *filtration* is a collection of  $\sigma$ -algebras  $(\mathcal{F}_t)_{t \geq 0}$  such that  $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$  if  $0 \leq s < t$  [49].

**Definition 3.2.7.** A stochastic process  $(X_t)_{t \geq 0}$  is *adapted to the filtration*  $(\mathcal{F}_t)_{t \geq 0}$  if, for

each  $t$ ,  $X_t$  is a  $\mathcal{F}_t$ -measurable random variable [49].

**Definition 3.2.8.** Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  be a filtered probability space. For each  $\omega \in \Omega$ , suppose there is a continuous function  $B_t(\omega)$  of  $t \geq 0$  that satisfies  $B_0(\omega) = 0$ . Then, the stochastic process  $(B_t)_{t \geq 0}$  is a *Brownian motion* if for all  $0 = t_0 < t_1 < \dots < t_m$  the increments

$$B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_m} - B_{t_{m-1}} \quad (3.6)$$

are independent and normally distributed with [49]

$$\mathbb{E} [B_{t_{i+1}} - B_{t_i}] = 0 \quad (3.7)$$

$$\text{Var} [B_{t_{i+1}} - B_{t_i}] = t_{i+1} - t_i. \quad (3.8)$$

**Definition 3.2.9.** Let  $B_t$ ,  $t \geq 0$ , be a Brownian motion and let  $(\mathcal{F}_t)_{t \geq 0}$  be an associated filtration. An *Itô (drift-diffusion) process* is a stochastic process of the form

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dB_s, \quad (3.9)$$

where  $X_0$  is non-random and  $(\mu_s)_{s \geq 0}$  and  $(\sigma_s)_{s \geq 0}$  are adapted stochastic processes [49].

The formula (3.9) can also be written in differential form as

$$dX_t = \mu_t dt + \sigma_t dB_t. \quad (3.10)$$

**Lemma 3.2.10** (Itô-Dôebelin formula for an Itô process). Let  $(X_t)_{t \geq 0}$  be an Itô process and let  $f(x, t)$  be a function for which the partial derivatives  $f_t(x, t)$ ,  $f_x(x, t)$ , and  $f_{xx}(x, t)$  are continuous. Then, for every  $t \geq 0$ ,

$$df(X_t, t) = f_t(X_t, t) dt + \mu_t f_x(X_t, t) dt + \frac{1}{2} \sigma_t^2 f_{xx}(X_t, t) dt + \sigma_t f_x(X_t, t) dB_t \quad (3.11)$$



The result above is also known as Itô's lemma [49].

**Definition 3.2.11.** Let  $\mu > 0$  and  $\sigma > 0$  be constants. *Geometric Brownian motion* is a continuous stochastic process  $(S_t)_{t \geq 0}$  which satisfies the SDE [31]

$$dS_t = \mu S_t dt + \sigma S_t dB_t. \quad (3.12)$$

Here,  $\mu$  is called the drift and  $\sigma$  is called the volatility of the geometric Brownian motion  $(S(t))_{t \geq 0}$ . Using Itô's lemma on  $f(x, t) = \log(x)$ , we get [49]

$$S_t = S_0 \exp \left\{ \sigma B_t + \left( \mu - \frac{1}{2} \sigma^2 \right) t \right\}. \quad (3.13)$$

Then,  $S_t$  has mean [49]

$$\mathbb{E}[S_t] = S_0 e^{\mu t}, \quad (3.14)$$

and variance

$$\begin{aligned} \text{Var}(S_t) &= \mathbb{E}[S_t^2] - S_0^2 e^{2\mu t} \\ &= \mathbb{E} \left[ S_0^2 \exp \left\{ 2\sigma B_t + 2 \left( \mu - \frac{1}{2} \sigma^2 \right) t \right\} \right] - S_0^2 e^{2\mu t} \\ &= S_0^2 \left( e^{2\mu t + \sigma^2 t} - e^{2\mu t} \right) \\ &= S_0^2 e^{2\mu t} \left( e^{\sigma^2 t} - 1 \right). \end{aligned} \quad (3.15)$$

The probability density function of  $S_t$  is given by [49]

$$f_t(x) = \frac{1}{\sigma x \sqrt{2\pi t}} \exp \left\{ -\frac{(\ln x - (\mu - \frac{1}{2} \sigma^2) t - \ln S_0)^2}{2\sigma^2 t} \right\}, \quad x > 0. \quad (3.16)$$

**Definition 3.2.12.** Let  $T$  be a fixed positive number. An adapted stochastic process  $(M_t)_{t \in [0, T]}$ , is a *martingale with respect to filtration*  $(\mathcal{F}_t)_{t \in [0, t]}$  under a probability measure  $\mathbb{Q}$  if [49]:

- (i)  $\mathbb{E}^{\mathbb{Q}} [|M_t|] < \infty$  for all  $t \in [0, T]$ ;
- (ii)  $M_t$  is  $\mathcal{F}_t$  - measurable;
- (iii)  $\mathbb{E}^{\mathbb{Q}} [M_t | \mathcal{F}_s] = M(s)$  for all  $t \in [0, T]$  .

**Definition 3.2.13.** A *risk-neutral (martingale) measure*  $\mathbb{Q}$  is a probability measure equivalent to  $\mathbb{P}$  such that the discounted asset price process  $(e^{-rt}S_t)_{t \in [0, T]}$  (if the risk-free interest rate  $r$  is a constant) is a martingale under  $\mathbb{Q}$  [49].

**Definition 3.2.14.** If  $X$  is an integrable random variable and  $\mathcal{G} \subset \mathcal{F}$  is a  $\sigma$ -algebra, then there exists a random variable  $\mathbb{E}[X|\mathcal{G}]$ , called the *conditional expectation of  $X$  given  $\mathcal{G}$* , where  $\mathbb{E}[X|\mathcal{G}]$  is  $\mathcal{G}$ -measurable, integrable and satisfies

$$\int_G \mathbb{E}[X|\mathcal{G}] d\mathbb{P} = \int_G X d\mathbb{P} \quad (3.17)$$

for all  $G \in \mathcal{G}$  [12].

### 3.3 Modelling an Ijārah Sukūk Call Option Price

In this section we derive the pricing formula for the mid-term call option of Ijārah Sukūk. Using the fact that the Sukūk call option can also be exercised at mid-term, an appropriate condition is needed which takes care of this choice. The option should be exercised at mid-term if the payoff at mid-term is more than the expected value of the payoff at expiry discounted at mid-term minus the Ijārah payments that already paid till mid-term. After taking the Ijārah payments in consideration, this gives us the required condition to exercise at mid-term. It can be written as a measurable set and can be simplified further by computing the expected value as an integral. Taking into consideration this condition, the price of the option is then derived by computing the expectation of the discounted payoff under the risk-neutral measure.

### 3.3.1 Statement of the Problem

European options are exercised only on maturity. However mid-term exercising of the call option is possible in Ijārah Sukūk. Therefore, we will study how the valuation formula changes under such circumstances using both analytical derivation and numerical simulation, and introduce the mathematical model of a mid-term exercise option in this chapter to demonstrate the pricing possibilities of this idea. It means that we want to see the possibility of adjusting the time factor to accommodate this scenario. Moreover, we will compare the price of Sukūk call option with the prices of the corresponding European and American call options for life-span.

**Definition 3.3.1.** The buyer of the *call option* has the right but not the obligation to use the option to buy the underlying assets at mid-term of maturity,  $T/2$  or at expiry  $T$ .

### 3.3.2 Early Exercise

- For Sukūk call option, there is some chance that Sukūk option will be exercised early at mid-term  $\frac{T}{2}$
- For European call option, the exercise is only at maturity  $T$ .
- For American call option, the exercise can be carried out at any time over the life-span.

### 3.3.3 Assumptions and Notation

- (a) The price of the asset follows a geometric Brownian motion  $S_t$ ,

$$S_t = S_0 \exp \left\{ \sigma B_t + \left( \mu - \frac{1}{2} \sigma^2 \right) t \right\}. \quad (3.18)$$

- (b) The duration of the contract is  $T$ .
- (c)  $(B_t)_{t \geq 0}$  is a standard Brownian motion.
- (d)  $\mu$  is the drift of the geometric Brownian motion.
- (e)  $\sigma$  is the volatility of the geometric Brownian motion.
- (f)  $r$  is the risk-free Ijārah rate which can be fixed using the Islamic interbank benchmark rate, IIBR, as benchmark (see Section 2.1.3).
- (g)  $K$  is the strike price which a derivative contract can be exercised.
- (h)  $q$  is Ijārah payments which is the return to the Sukūk holder from the underlying assets.
- (i) The call option can be exercised at the mid-term  $T/2$  (or  $t_{\frac{n}{2}}$ ) or at  $T$ .
- (j)  $(\mathcal{F}_t)_{t \geq 0}$  is the filtration.
- (k) There exists an equivalent martingale measure  $\mathbb{Q}$  such that the discounted asset price  $\tilde{S}_t = e^{-rt}S_t$  becomes a martingale under  $\mathbb{Q}$  and so  $\mathbb{E}^{\mathbb{Q}}[e^{-rT}S_T] = S_0$ .
- (l) No transactions cost.

### 3.3.4 Condition for the Buyer of the Call Option to Exercise it at Mid-Term

The important question and mathematical challenge is how the buyer of the call option exercises the option and what is the optimal condition to exercise at mid-term instead of expiry time  $T$ . It is assumed that the buyer of the call option will do so only if he can make more profit or suffer less loss than he does when exercise the option at the

expiry time  $T$ , i.e. if the buyer of the call option exercises the option at mid-term, then the following condition should be satisfied:

$$\left(S_{t_{\frac{n}{2}}} - K\right)^+ \geq \mathbb{E} \left[ (S_T - K)^+ e^{-\frac{rT}{2}} \middle| \mathcal{F}_{t_{\frac{n}{2}}} \right] - \sum_{i=1}^{\frac{n}{2}} q e^{-\frac{rTi}{n}}, \quad (3.19)$$

where  $q$  is the periodic distribution amount (the Ijārah payment) which is considered as income return to the Sukūk holder from the underlying assets or the same coupon payments in western bond, where  $n$  is the number of periods of payments. To explain the above condition, the left-hand side is the payoff of the call option at mid-term while the right-hand side is the discounted expected payoff of the call option conditional to the  $\sigma$ -algebra  $\mathcal{F}_{t_{\frac{n}{2}}}$ . Therefore, this is the condition that when the buyer of the call option should exercise it. Let  $A$  be the set where this is true, i.e.

$$A = \left\{ \omega \in \Omega : \left(S_{t_{\frac{n}{2}}} - K\right)^+ \geq \mathbb{E} \left[ (S_T - K)^+ e^{-\frac{rT}{2}} \middle| \mathcal{F}_{t_{\frac{n}{2}}} \right] - \sum_{i=1}^{\frac{n}{2}} q e^{-\frac{rTi}{n}} \right\}. \quad (3.20)$$

Then  $A \in \mathcal{F}$  because both  $S_{t_{\frac{n}{2}}}$  and  $\mathbb{E} \left[ (S_T - K)^+ e^{-\frac{rT}{2}} \middle| \mathcal{F}_{t_{\frac{n}{2}}} \right]$  being random variables are measurable.

As known in above assumptions, the price of call option is the discounted expected payoff under measure  $\mathbb{Q}$ . Therefore, the price of the call option of Sukūk based on the above condition can be decomposed into

$$C_S = e^{-\frac{rT}{2}} \mathbb{E}^{\mathbb{Q}} \left[ \left(S_{t_{\frac{n}{2}}} - K\right)^+ | A \right] + e^{-rT} \mathbb{E}^{\mathbb{Q}} \left[ (S_T - K)^+ | A^c \right]. \quad (3.21)$$

### 3.3.5 Simplification of the Condition

Now we are going to simplify the condition defined in (3.19). By the assumption (a) in Section 3.3.3, the process  $(S_t)_{t \geq 0}$  is a solution of the SDE (3.12) and we apply (3.11)

(Itô's Lemma Formula) to find the expectation (3.14) and probability density function (3.16).

The expectation in the condition to exercise at mid-term (3.19) is the conditional expectation with respect to sigma algebra  $\mathcal{F}_{t_{\frac{n}{2}}}$ , where the time horizon becomes  $[\frac{T}{2}, T]$ . Since  $S_T$  follows log-normal distribution, using the probability density function given in (3.16) by replacing  $S_0$  by  $S_{t_{\frac{n}{2}}}$  and  $t$  by  $\frac{T}{2}$ , we have

$$\begin{aligned}
& \mathbb{E} \left[ (S_T - K)^+ \mid \mathcal{F}_{t_{\frac{n}{2}}} \right] \\
&= \int_K^\infty x \frac{1}{\sigma x \sqrt{2\pi \frac{T}{2}}} \exp \left\{ -\frac{\left( \ln x - \left( \mu - \frac{1}{2}\sigma^2 \right) \frac{T}{2} - \ln S_{t_{\frac{n}{2}}} \right)^2}{2\sigma^2 \frac{T}{2}} \right\} dx \\
&\quad - \int_K^\infty K \frac{1}{\sigma x \sqrt{2\pi \frac{T}{2}}} \exp \left\{ -\frac{\left( \ln x - \left( \mu - \frac{1}{2}\sigma^2 \right) \frac{T}{2} - \ln S_{t_{\frac{n}{2}}} \right)^2}{2\sigma^2 \frac{T}{2}} \right\} dx \\
&= \int_{\ln K}^\infty x \frac{1}{\sigma \sqrt{\pi T}} \exp \left\{ -\frac{\left( \ln x - \ln S_{t_{\frac{n}{2}}} - \frac{T}{2}\mu + \frac{1}{4}\sigma^2 T \right)^2}{\sigma^2 T} \right\} d(\ln x) \\
&\quad - \int_{\ln K}^\infty K \frac{1}{\sigma \sqrt{\pi T}} \exp \left\{ -\frac{\left( \ln x - \ln S_{t_{\frac{n}{2}}} - \frac{T}{2}\mu + \frac{1}{4}\sigma^2 T \right)^2}{\sigma^2 T} \right\} d(\ln x). \tag{3.22}
\end{aligned}$$

We will make a change in variables so that

$$y = \ln(x) - \ln S_{t_{\frac{n}{2}}} - \frac{T}{2}\mu + \frac{1}{4}\sigma^2 T, \tag{3.23}$$

we have

$$x = \exp \left\{ y + \ln S_{t_{\frac{n}{2}}} + \frac{T}{2}\mu - \frac{1}{4}\sigma^2 T \right\}, \tag{3.24}$$

and

$$dx = \exp \left\{ y + \ln S_{t_{\frac{n}{2}}} + \frac{T}{2}\mu - \frac{1}{4}\sigma^2 T \right\} dy, \tag{3.25}$$

where the Jacobian is

$$\exp \left\{ y + \ln S_{t_{\frac{n}{2}}} + \frac{T}{2}\mu - \frac{1}{4}\sigma^2 T \right\}, \quad (3.26)$$

Substituting into (3.22), we get

$$\begin{aligned} & \mathbb{E} \left[ (S_T - K)^+ \mid \mathcal{F}_{t_{\frac{n}{2}}} \right] \\ &= \int_{\ln K - \ln S_{t_{\frac{n}{2}}} - \frac{T}{2}\mu + \frac{1}{4}\sigma^2 T}^{\infty} \frac{1}{\sigma\sqrt{\pi T}} e^{y + \ln S_{t_{\frac{n}{2}}} + \frac{T}{2}\mu - \frac{1}{4}\sigma^2 T} e^{-\frac{y^2}{\sigma^2 T}} dy \\ & \quad - K \int_{\ln K - \ln S_{t_{\frac{n}{2}}} - \frac{T}{2}\mu + \frac{1}{4}\sigma^2 T}^{\infty} \frac{1}{\sigma\sqrt{\pi T}} e^{-\frac{y^2}{\sigma^2 T}} dy. \end{aligned} \quad (3.27)$$

Completing the square and then manipulating the first exponential to make it density of normal distribution form, we have

$$\begin{aligned} \mathbb{E} \left[ (S_T - K)^+ \mid \mathcal{F}_{t_{\frac{n}{2}}} \right] &= S_{t_{\frac{n}{2}}} e^{\frac{T}{2}\mu} \int_{\ln K - \ln S_{t_{\frac{n}{2}}} - \frac{T}{2}\mu + \frac{1}{4}\sigma^2 T}^{\infty} \frac{1}{\sigma\sqrt{\pi T}} e^{-\frac{(y - \frac{1}{2}\sigma^2 T)^2}{\sigma^2 T}} dy \\ & \quad - K \int_{\ln K - \ln S_{t_{\frac{n}{2}}} - \frac{T}{2}\mu + \frac{1}{4}\sigma^2 T}^{\infty} \frac{1}{\sigma\sqrt{\pi T}} e^{-\frac{y^2}{\sigma^2 T}} dy \\ &= S_{t_{\frac{n}{2}}} e^{\frac{T}{2}\mu} \int_{\ln K - \ln S_{t_{\frac{n}{2}}} - \frac{T}{2}\mu + \frac{1}{4}\sigma^2 T}^{\infty} \frac{1}{\sigma\sqrt{\pi T}} e^{-\frac{\left(\frac{y - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T/2}}\right)^2}{2}} dy \\ & \quad - K \int_{\ln K - \ln S_{t_{\frac{n}{2}}} - \frac{T}{2}\mu + \frac{1}{4}\sigma^2 T}^{\infty} \frac{1}{\sigma\sqrt{\pi T}} e^{-\frac{y^2}{\sigma^2 T}} dy. \end{aligned} \quad (3.28)$$

Substituting  $z = \frac{y}{\sigma\sqrt{\frac{T}{2}}} - \sigma\sqrt{\frac{T}{2}}$  in the first integral and  $w = \frac{y}{\sigma\sqrt{\frac{T}{2}}}$  in the second, we have

$$\begin{aligned} \mathbb{E} \left[ (S_T - K)^+ \mid \mathcal{F}_{t_{\frac{n}{2}}} \right] &= S_{t_{\frac{n}{2}}} e^{\frac{T}{2}\mu} \int_{\frac{\ln K - \ln S_{t_{\frac{n}{2}}} - \frac{T}{2}\mu + \frac{1}{4}\sigma^2 T}{\sigma\sqrt{\frac{T}{2}}} - \sigma\sqrt{\frac{T}{2}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\ & \quad - K \int_{\frac{\ln K - \ln S_{t_{\frac{n}{2}}} - \frac{T}{2}\mu + \frac{1}{4}\sigma^2 T}{\sigma\sqrt{\frac{T}{2}}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} dw \\ &= S_{t_{\frac{n}{2}}} e^{\frac{T}{2}\mu} \int_{-d_1(T)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz - K \int_{-d_2(T)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} dw \end{aligned}$$

$$= S_{t_{\frac{n}{2}}} e^{\frac{T}{2}\mu} \Phi(d_1(T)) - K \Phi(d_2(T)), \quad (3.29)$$

where

$$d_1(T) = \frac{\ln S_{t_{\frac{n}{2}}} + \frac{T}{2}\mu + \frac{1}{4}\sigma^2 T - \ln K}{\sigma \sqrt{\frac{T}{2}}} \quad (3.30)$$

$$d_2(T) = \frac{\ln S_{t_{\frac{n}{2}}} + \frac{T}{2}\mu - \frac{1}{4}\sigma^2 T - \ln K}{\sigma \sqrt{\frac{T}{2}}}, \quad (3.31)$$

and  $\Phi(d_1(T)) = \int_{-d_1(T)}^{\infty} \varphi(x) dx$  is the cumulative distribution of a standard normal (Gaussian) random variable (using the symmetry of the probability density to adjust the limits of integration), and  $\varphi(x)$  is the density of the standard normal random variable.

Thus, (3.29) allows condition (3.19) to be simplified to

$$\left(S_{t_{\frac{n}{2}}} - K\right)^+ \geq S_{t_{\frac{n}{2}}} e^{\frac{T}{2}\mu - \frac{1}{2}rT} \Phi(d_1(T)) - K e^{-\frac{1}{2}rT} \Phi(d_2(T)) - \sum_{i=1}^{\frac{n}{2}} q e^{-\frac{rTi}{n}}. \quad (3.32)$$

### 3.3.6 Derivations of Call Options Price Formulas

Next, we consider the price of the call option (3.21). Notice first that (3.32) is equivalent to

$$\left\{ 0 \leq S_{t_{\frac{n}{2}}} e^{\frac{T}{2}\mu - \frac{1}{2}rT} \Phi(d_1(T)) \leq K e^{-\frac{1}{2}rT} \Phi(d_2(T)) + \sum_{i=1}^{\frac{n}{2}} q e^{-\frac{rTi}{n}} \right\} \quad (3.33)$$

on  $\{S_{t_{\frac{n}{2}}} \leq K\}$  and

$$\left\{ S_{t_{\frac{n}{2}}} - K \geq S_{t_{\frac{n}{2}}} e^{\frac{T}{2}\mu - \frac{1}{2}rT} \Phi(d_1(T)) - K e^{-\frac{1}{2}rT} \Phi(d_2(T)) - \sum_{i=1}^{\frac{n}{2}} q e^{-\frac{rTi}{n}} > 0 \right\}, \quad (3.34)$$



on  $\{S_{t_{\frac{n}{2}}} > K\}$ . We set an appropriate  $K$  such that (3.32) is always satisfied. We claim that if

$$K \geq \frac{K - Ke^{-\frac{1}{2}rT}\Phi(d_2(T)) - \sum_{i=1}^{\frac{n}{2}} qe^{-\frac{rTi}{n}}}{1 - e^{\frac{T}{2}\mu - \frac{1}{2}rT}\Phi(d_1(T))}, \quad (3.35)$$

and  $1 - e^{\frac{T}{2}\mu - \frac{1}{2}rT}\Phi(d_1(T)) > 0$ , then (3.32) holds. This is because on  $\{S_{t_{\frac{n}{2}}} > K\}$ , we have

$$S_{t_{\frac{n}{2}}} > K \geq \frac{K - Ke^{-\frac{1}{2}rT}\Phi(d_2(T)) - \sum_{i=1}^{\frac{n}{2}} qe^{-\frac{rTi}{n}}}{1 - e^{\frac{T}{2}\mu - \frac{1}{2}rT}\Phi(d_1(T))}, \quad (3.36)$$

which is equivalent to (3.34). Using  $1 - e^{\frac{T}{2}\mu - \frac{1}{2}rT}\Phi(d_1(T)) > 0$  in (3.35), we get

$$\frac{Ke^{-\frac{1}{2}rT}\Phi(d_2(T)) + \sum_{i=1}^{\frac{n}{2}} qe^{-\frac{rTi}{n}}}{e^{\frac{T}{2}\mu - \frac{1}{2}rT}\Phi(d_1(T))} \geq K. \quad (3.37)$$

On  $\{S_{t_{\frac{n}{2}}} \leq K\}$ , (3.37) implies that (3.33) is satisfied. We assume that we can choose  $K$  such that (3.35) is satisfied. We apply a change of measure to obtain the asset price under the risk- neutral measure  $\mathbb{Q}$ . Therefore, the asset price satisfies

$$dS_t = rS_t dt + \sigma S_t d\widetilde{B}_t \quad (3.38)$$

where  $d\widetilde{B}_t = dB_t + \theta dt$  and  $\theta = \frac{\mu-r}{\sigma}$  is the market price of risk, and  $\widetilde{B}_t$  is a Brownian motion under the risk-neutral measure  $\mathbb{Q}$ , for which existence and uniqueness is guaranteed by Girsanov's theorem [49]. The asset price  $S_t$  is not a martingale under  $\mathbb{Q}$ . Let  $\widetilde{S}_t = e^{-rt}S_t$  be the discounted asset price, so Itô's lemma yields the SDE  $d\widetilde{S}_t = \sigma\widetilde{S}_t d\widetilde{B}_t$ .

By the choice of  $K$  given in (3.36), the condition (3.32) is always satisfied, and,

therefore, the first part of the call option price (3.21) becomes

$$\begin{aligned}
& e^{-\frac{rT}{2}} \mathbb{E}^{\mathbb{Q}} \left[ \left( S_{t_{\frac{n}{2}}} - K \right)^+ \middle| A \right] \\
&= e^{-\frac{rT}{2}} \mathbb{E}^{\mathbb{Q}} \left[ \left( S_{t_{\frac{n}{2}}} - K \right)^+ \right] \\
&= \int_{Ke^{-r\frac{T}{2}}}^{\infty} \left( x - Ke^{-r\frac{T}{2}} \right) \frac{1}{\sigma x \sqrt{2\pi\frac{T}{2}}} \exp \left[ -\frac{(\ln x - \ln S_0 + \frac{1}{4}\sigma^2 T)^2}{2\sigma^2\frac{T}{2}} \right] dx \\
&= \int_{\ln(Ke^{-r\frac{T}{2}})}^{\infty} x \frac{1}{\sigma \sqrt{\pi T}} \exp \left[ -\frac{(\ln x - \ln S_0 + \frac{1}{4}\sigma^2 T)^2}{\sigma^2 T} \right] d(\ln x) \\
&\quad - Ke^{-r\frac{T}{2}} \int_{\ln(Ke^{-r\frac{T}{2}})}^{\infty} \frac{1}{\sigma \sqrt{\pi T}} \exp \left[ -\frac{(\ln x - \ln S_0 + \frac{1}{4}\sigma^2 T)^2}{\sigma^2 T} \right] d(\ln x). \tag{3.39}
\end{aligned}$$

We will make a change in variables so that

$$y = \ln(x) - \ln(S_0) + \frac{1}{4}\sigma^2 T. \tag{3.40}$$

We then have

$$x = \exp \left\{ y + \ln(S_0) - \frac{1}{4}\sigma^2 T \right\}, \tag{3.41}$$

and

$$dx = \exp \left\{ y + \ln(S_0) - \frac{1}{4}\sigma^2 T \right\} dy, \tag{3.42}$$

where the Jacobian is

$$\exp \left\{ y + \ln(S_0) - \frac{1}{4}\sigma^2 T \right\}. \tag{3.43}$$

Substituting into (3.39), we get

$$e^{-\frac{rT}{2}} \mathbb{E}^{\mathbb{Q}} \left[ \left( S_{t_{\frac{n}{2}}} - K \right)^+ \middle| A \right]$$

$$\begin{aligned}
&= \int_{\ln(K) - \frac{1}{2}rT - \ln S_0 + \frac{1}{4}\sigma^2 T}^{\infty} \frac{1}{\sigma\sqrt{\pi T}} e^{y + \ln S_0 - \frac{1}{4}\sigma^2 T} e^{-\frac{y^2}{\sigma^2 T}} dy \\
&\quad - K e^{-r\frac{T}{2}} \int_{\ln(K) - \frac{1}{2}rT - \ln S_0 + \frac{1}{4}\sigma^2 T}^{\infty} \frac{1}{\sigma\sqrt{\pi T}} e^{-\frac{y^2}{\sigma^2 T}} dy. \quad (3.44)
\end{aligned}$$

After completing the square in the exponential, this yields

$$\begin{aligned}
e^{-\frac{rT}{2}} \mathbb{E}^{\mathbb{Q}} \left[ \left( S_{t_{\frac{T}{2}}} - K \right)^+ \middle| A \right] &= S_0 \int_{\ln(K) - \frac{1}{2}rT - \ln S_0 + \frac{1}{4}\sigma^2 T}^{\infty} \frac{1}{\sigma\sqrt{\pi T}} e^{-\frac{(y - \frac{1}{2}\sigma^2 T)^2}{\sigma^2 T}} dy \\
&\quad - K e^{-r\frac{T}{2}} \int_{\ln(K) - \frac{1}{2}rT - \ln S_0 + \frac{1}{4}\sigma^2 T}^{\infty} \frac{1}{\sigma\sqrt{\pi T}} e^{-\frac{y^2}{\sigma^2 T}} dy. \quad (3.45)
\end{aligned}$$

Substituting  $z = \frac{y}{\sigma\sqrt{\frac{T}{2}}} - \sigma\sqrt{\frac{T}{2}}$  in the first integral and  $w = \frac{y}{\sigma\sqrt{\frac{T}{2}}}$  in the second, we have

$$\begin{aligned}
&e^{-\frac{rT}{2}} \mathbb{E}^{\mathbb{Q}} \left[ \left( S_{t_{\frac{T}{2}}} - K \right)^+ \middle| A \right] \\
&= S_0 \int_{\frac{\ln K - \frac{1}{2}rT - \ln S_0 + \frac{1}{4}\sigma^2 T}{\sigma\sqrt{\frac{T}{2}}} - \sigma\sqrt{\frac{T}{2}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\
&\quad - K e^{-r\frac{T}{2}} \int_{\frac{\ln K - \frac{1}{2}rT - \ln S_0 + \frac{1}{4}\sigma^2 T}{\sigma\sqrt{\frac{T}{2}}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} dw \\
&= S_0 \int_{-d_3(\frac{T}{2})}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz - K e^{-r\frac{T}{2}} \int_{-d_4(\frac{T}{2})}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} dw \\
&= S_0 \Phi \left( d_3 \left( \frac{T}{2} \right) \right) - K e^{-r\frac{T}{2}} \Phi \left( d_4 \left( \frac{T}{2} \right) \right), \quad (3.46)
\end{aligned}$$

where

$$d_3 \left( \frac{T}{2} \right) = \frac{\ln(S_0/K) + (r + \frac{1}{2}\sigma^2) \cdot \frac{T}{2}}{\sigma\sqrt{\frac{T}{2}}}, \quad (3.47)$$

$$d_4 \left( \frac{T}{2} \right) = \frac{\ln(S_0/K) + (r - \frac{1}{2}\sigma^2) \cdot \frac{T}{2}}{\sigma\sqrt{\frac{T}{2}}}. \quad (3.48)$$

Next, we simplify the second part of (3.21),

$$\begin{aligned} & e^{-rT} \mathbb{E}^{\mathbb{Q}} [(S_T - K)^+ | A^c] \\ &= \mathbb{E}^{\mathbb{Q}} \left[ \left( \widetilde{S}_T - K e^{-rT} \right)^+ \middle| \left\{ S_{t_{\frac{n}{2}}} > \left( K e^{-r\frac{T}{2}} \Phi(d_2(T)) + \sum_{i=1}^{\frac{n}{2}} q e^{-\frac{rTi}{n}} \right) \frac{e^{-\frac{T}{2}\mu + \frac{1}{2}rT}}{\Phi(d_1(T))} \right\} \right]. \end{aligned}$$

Using Itô's Lemma on the function  $\ln x$ , we get  $d(\ln \widetilde{S}_t) = -\frac{1}{2}\sigma^2 dt + \sigma d\widetilde{B}_t$  and integrating between  $T/2$  and  $T$  yields

$$\widetilde{S}_T = \widetilde{S}_{t_{\frac{n}{2}}} \exp \left( -\frac{1}{4}\sigma^2 T + \sigma (\widetilde{B}_T - \widetilde{B}_{\frac{T}{2}}) \right). \quad (3.49)$$

Therefore,

$$\begin{aligned} & e^{-rT} \mathbb{E}^{\mathbb{Q}} [(S_T - K)^+ | A^c] \\ &= \mathbb{E}^{\mathbb{Q}} \left[ \left( \widetilde{S}_{t_{\frac{n}{2}}} e^{-\frac{1}{4}\sigma^2 T + \sigma (\widetilde{B}_T - \widetilde{B}_{\frac{T}{2}})} - K e^{-rT} \right)^+ \middle| B \right], \end{aligned}$$

where

$$B = \left\{ \widetilde{S}_{t_{\frac{n}{2}}} > \left( K e^{-r\frac{T}{2}} \Phi(d_2(T)) + \sum_{i=1}^{\frac{n}{2}} q e^{\frac{rTi}{n}} \right) \frac{e^{-\frac{T}{2}\mu}}{\Phi(d_1(T))} \right\}. \quad (3.50)$$

Since  $K e^{-rT} < K e^{-\frac{1}{2}rT}$ ,

$$e^{-rT} \mathbb{E}^{\mathbb{Q}} [(S_T - K)^+ | A^c] = \mathbb{E}^{\mathbb{Q}} \left[ \left( \widetilde{S}_{t_{\frac{n}{2}}} e^{-\frac{1}{4}\sigma^2 T + \sigma (\widetilde{B}_T - \widetilde{B}_{\frac{T}{2}})} - K e^{-rT} \right)^+ \right]. \quad (3.51)$$

Let  $Z = (\widetilde{B}_T - \widetilde{B}_{\frac{T}{2}})$  and  $f_Z$  be the probability density of  $Z$ . Since  $Z$  is independent of  $\widetilde{S}_{t_{\frac{n}{2}}}$ , so

$$e^{-rT} \mathbb{E}^{\mathbb{Q}} [(S_T - K)^+ | A^c] = \int_{-\infty}^{\infty} \mathbb{E}^{\mathbb{Q}} \left[ \left( \widetilde{S}_{t_{\frac{n}{2}}} e^{-\frac{1}{4}\sigma^2 T + \sigma \beta} - K e^{-rT} \right)^+ \right] f_Z(\beta) d\beta$$

$$= \int_{-\infty}^{\infty} \mathbb{E}^{\mathbb{Q}} \left[ \left( \widetilde{S_{t_{\frac{T}{2}}}} \alpha - K e^{-rT} \right)^+ \right] f_Z(\beta) d\beta, \quad (3.52)$$

where  $\alpha = e^{-\frac{1}{4}\sigma^2 T + \sigma\beta}$ . Since  $Z = \widetilde{B_T} - \widetilde{B_{\frac{T}{2}}}$  is the increment of Brownian motion and normal with mean 0 and variance  $\frac{T}{2}$ , then, if  $f(x)$  is the probability density function of  $\widetilde{S}$

$$\begin{aligned} e^{-rT} \mathbb{E}^{\mathbb{Q}} \left[ (S_T - K)^+ | A^c \right] &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \frac{T}{2}}} e^{-\frac{1}{2} \frac{\beta^2}{T/2}} \mathbb{E}^{\mathbb{Q}} \left[ \left( \widetilde{S_{t_{\frac{T}{2}}}} \alpha - K e^{-rT} \right)^+ \right] d\beta \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi T}} e^{-\frac{\beta^2}{T}} \left( \int_{\frac{K e^{-rT}}{\alpha}}^{\infty} (\alpha x - K e^{-rT}) f_{T/2}(x) dx \right) d\beta. \end{aligned} \quad (3.53)$$

The inner integral with respect to  $x$  becomes

$$\begin{aligned} \int_{\frac{K e^{-rT}}{\alpha}}^{\infty} (\alpha x - K e^{-rT}) f_{T/2}(x) dx \\ = \int_{K e^{-rT} + \frac{1}{4}\sigma^2 T - \sigma\beta}^{\infty} \frac{x e^{-\frac{1}{4}\sigma^2 T + \sigma\beta} - K e^{-rT}}{x \sigma \sqrt{\pi T}} \exp \left[ -\frac{(\ln x - \ln S_0 + \frac{1}{4}\sigma^2 T)^2}{\sigma^2 T} \right] dx. \end{aligned}$$

Substituting

$$y = \ln x - \ln S_0 + \frac{1}{4}\sigma^2 T, \quad (3.54)$$

so that

$$x = \exp \left\{ y + \ln S_0 - \frac{1}{4}\sigma^2 T \right\}, \quad (3.55)$$

and

$$dx = \exp \left\{ y + \ln S_0 - \frac{1}{4}\sigma^2 T \right\} dy, \quad (3.56)$$

where the Jacobian is

$$\exp \left\{ y + \ln S_0 - \frac{1}{4} \sigma^2 T \right\}. \quad (3.57)$$

So, we have

$$\begin{aligned} \int_{\frac{Ke^{-rT}}{\alpha}}^{\infty} (\alpha x - Ke^{-rT}) f_{T/2}(x) dx \\ = \int_{\ln K - rT + \frac{1}{2} \sigma^2 T - \ln S_0 - \sigma \beta}^{\infty} \frac{e^{y + \ln S_0 - \frac{1}{4} \sigma^2 T} e^{-\frac{1}{4} \sigma^2 T + \sigma \beta} - Ke^{-rT}}{\sigma \sqrt{\pi T}} e^{-\frac{y^2}{\sigma^2 T}} dy. \end{aligned}$$

Completing the square, we get

$$\begin{aligned} \int_{\frac{Ke^{-rT}}{\alpha}}^{\infty} (\alpha x - Ke^{-rT}) f_{T/2}(x) dx \\ = S_0 e^{-\frac{1}{4} \sigma^2 T + \sigma \beta} \int_{\ln K - rT - \ln S_0 + \frac{1}{2} \sigma^2 T - \sigma \beta}^{\infty} \frac{1}{\sigma \sqrt{\pi T}} e^{-\frac{(y - \frac{1}{2} \sigma^2 T)^2}{\sigma^2 T}} dy \\ - Ke^{-rT} \int_{\ln K - rT - \ln S_0 + \frac{1}{2} \sigma^2 T - \sigma \beta}^{\infty} \frac{1}{\sigma \sqrt{\pi T}} e^{-\frac{y^2}{\sigma^2 T}} dy. \end{aligned} \quad (3.58)$$

Then, substituting  $z = \frac{y - \frac{1}{2} \sigma^2 T}{\sigma \sqrt{\frac{T}{2}}}$  and  $w = \frac{y}{\sigma \sqrt{\frac{T}{2}}}$  in the resulting two integrals, we have

$$\begin{aligned} \int_{\frac{Ke^{-rT}}{\alpha}}^{\infty} (\alpha x - Ke^{-rT}) f_{T/2}(x) dx \\ = S_0 e^{-\frac{1}{4} \sigma^2 T + \sigma \beta} \int_{-d_5(\beta, T)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz - Ke^{-rT} \int_{-d_6(\beta, T)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} dw \\ = S_0 e^{-\frac{1}{4} \sigma^2 T + \sigma \beta} \Phi(d_5(\beta, T)) - Ke^{-rT} \Phi(d_6(\beta, T)), \end{aligned} \quad (3.59)$$

where

$$d_5(\beta, T) = \frac{\ln(S_0/K) + rT + \sigma \beta}{\sigma \sqrt{\frac{T}{2}}}, \quad (3.60)$$

$$d_6(\beta, T) = \frac{\ln(S_0/K) + (r - \frac{1}{2} \sigma^2)T + \sigma \beta}{\sigma \sqrt{\frac{T}{2}}}. \quad (3.61)$$

Substituting (3.59) in (3.53), we get

$$\begin{aligned}
& e^{-rT} \mathbb{E}^{\mathbb{Q}} [(S_T - K)^+ | A^c] \\
&= S_0 \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi T}} e^{-\frac{\beta^2}{T}} e^{-\frac{1}{4}\sigma^2 T + \sigma\beta} \Phi(d_5(\beta, T)) d\beta \\
&\quad - K e^{-rT} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi T}} e^{-\frac{\beta^2}{T}} \Phi(d_6(\beta, T)) d\beta.
\end{aligned} \tag{3.62}$$

Therefore, we can write the second part of the price of the call option as

$$\begin{aligned}
& e^{-rT} \mathbb{E}^{\mathbb{Q}} [(S_T - K)^+ | A^c] \\
&= S_0 \mathbf{\Lambda}_1(T) - K e^{-rT} \mathbf{\Lambda}_2(T),
\end{aligned} \tag{3.63}$$

where

$$\begin{aligned}
\mathbf{\Lambda}_1(T) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi T}} e^{-\frac{\beta^2}{T}} e^{-\frac{1}{4}\sigma^2 T + \sigma\beta} \Phi(d_5(\beta, T)) d\beta \\
&= \frac{1}{\pi\sqrt{2T}} \int_{-\infty}^{\infty} \int_{-d_5(\beta, T)}^{\infty} e^{-\frac{\beta^2}{T} - \frac{1}{4}\sigma^2 T + \sigma\beta - \frac{z^2}{2}} dz d\beta,
\end{aligned} \tag{3.64}$$

$$\begin{aligned}
\mathbf{\Lambda}_2(T) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi T}} e^{-\frac{\beta^2}{T}} \Phi(d_6(\beta, T)) d\beta \\
&= \frac{1}{\pi\sqrt{2T}} \int_{-\infty}^{\infty} \int_{-d_6(\beta, T)}^{\infty} e^{-\frac{\beta^2}{T} - \frac{w^2}{2}} dw d\beta.
\end{aligned} \tag{3.65}$$

Combining (3.46) and (3.63), we find a formula for the price of the Sukūk call option,

$$C_S(0) = \left[ S_0 \Phi \left( d_3 \left( \frac{T}{2} \right) \right) - K e^{-r\frac{T}{2}} \Phi \left( d_4 \left( \frac{T}{2} \right) \right) \right] + [S_0 \mathbf{\Lambda}_1(T) - K e^{-rT} \mathbf{\Lambda}_2(T)]. \tag{3.66}$$

We will later use (3.66) in simulations to compare the price of Sukūk call option at  $t = 0$  with European call option price determined by the Black-Scholes formula.

### 3.3.7 Mid-term Call Option of Sukūk at Time $t$

To use (3.66) to evaluate the Sukūk call option during the life-span, we will follow the same steps of the above simplifications by assuming  $\tau_s = \frac{T}{2} - t$  where  $0 \leq t \leq \frac{T}{2}$  and  $\tau = T - t$  where  $0 \leq t \leq T$  to show the discounted time during the life-span, so analogous to (3.66), we can derive the formula for price of mid-term call option of Sukūk at time  $t$  given by

$$C_S(t) = \mathbb{1}_{\{t \leq \frac{T}{2}\}} [S_0 \Phi(d_3(\tau_s)) - K e^{-r\tau_s} \Phi(d_4(\tau_s))] + [S_0 \Lambda_1(\tau) - K e^{-r\tau} \Lambda_2(\tau)], \quad (3.67)$$

where, in this case,

$$d_3(\tau_s) = \frac{\ln(S_0/K) + (r + \frac{1}{2}\sigma^2) \cdot \tau_s}{\sigma \sqrt{\tau_s}}, \quad (3.68)$$

$$d_4(\tau_s) = \frac{\ln(S_0/K) + (r - \frac{1}{2}\sigma^2) \cdot \tau_s}{\sigma \sqrt{\tau_s}}, \quad (3.69)$$

$$d_5(\beta, \tau) = \frac{\ln(S_0/K) + r\tau + \sigma\beta}{\sigma \sqrt{\frac{\tau}{2}}}, \quad (3.70)$$

$$d_6(\beta, \tau) = \frac{\ln(S_0/K) + (r - \frac{1}{2}\sigma^2)\tau + \sigma\beta}{\sigma \sqrt{\frac{\tau}{2}}}, \quad (3.71)$$

$$\begin{aligned} \Lambda_1(\tau) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi\tau}} e^{-\frac{\beta^2}{\tau}} e^{-\frac{1}{4}\sigma^2\tau + \sigma\beta} \Phi(d_5(\beta, \tau)) d\beta \\ &= \frac{1}{\pi\sqrt{2\tau}} \int_{-\infty}^{\infty} \int_{-d_5(\beta, \tau)}^{\infty} e^{-\frac{\beta^2}{\tau} - \frac{1}{4}\sigma^2\tau + \sigma\beta - \frac{z^2}{2}} dz d\beta, \end{aligned} \quad (3.72)$$

$$\begin{aligned} \Lambda_2(\tau) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi\tau}} e^{-\frac{\beta^2}{\tau}} \Phi(d_6(\beta, \tau)) d\beta \\ &= \frac{1}{\pi\sqrt{2\tau}} \int_{-\infty}^{\infty} \int_{-d_6(\beta, \tau)}^{\infty} e^{-\frac{\beta^2}{\tau} - \frac{w^2}{2}} dw d\beta. \end{aligned} \quad (3.73)$$

The indicator function  $\mathbb{1}_{\{t \leq \frac{T}{2}\}}$  in (3.67) is due to the fact that  $t > \frac{T}{2}$  implies that the Sukūk call option has not been exercised at mid-term and therefore, the first term in (3.67), which comes because of exercising at mid-term, is vanished and the price is given by the second term only.



### 3.3.8 Mid-term Call Option of Sukūk with Continuous Ijārah at Time $t$

Here, we assume a continuous Ijārah  $q$ . The expected return becomes  $r - q$  instead of  $r$  (i.e. with drift  $r - q$ ), which implies that the risk-neutral process for  $S_t$  follows

$$dS_t = (r - q)S_t dt + \sigma S_t dB_t. \quad (3.74)$$

Following the same derivation, as before,

$$S_t = S_0 \exp \left[ \left( r - q - \frac{1}{2}\sigma^2 \right) t + \sigma B_t \right], \quad (3.75)$$

where  $S_t$  still follows a log-normal distribution with the alternative drift of  $r - q$ . This implies the condition to exercise the option and using the same steps as the previous simplifications to find the Sukūk call option price with continuous Ijārah  $q$  at time  $t$ , which is

$$C_S(t) = \mathbb{1}_{\{t \leq \frac{T}{2}\}} \left[ S_0 e^{-q\tau_s} \Phi(d_{3q}(\tau_s)) - K e^{-r\tau_s} \Phi(d_{4q}(\tau_s)) \right] + \left[ S_0 e^{-q\tau} \Lambda_{1q}(\tau) - K e^{-r\tau} \Lambda_{2q}(\tau) \right], \quad (3.76)$$

where, in this case,

$$d_{3q}(\tau_s) = \frac{\ln(S_0/K) + (r - q + \frac{1}{2}\sigma^2) \cdot \tau_s}{\sigma \sqrt{\tau_s}}, \quad (3.77)$$

$$d_{4q}(\tau_s) = \frac{\ln(S_0/K) + (r - q - \frac{1}{2}\sigma^2) \cdot \tau_s}{\sigma \sqrt{\tau_s}}, \quad (3.78)$$

$$d_{5q}(\beta, \tau) = \frac{\ln(S_0/K) + (r - q) \cdot \tau + \sigma\beta}{\sigma \sqrt{\frac{\tau}{2}}}, \quad (3.79)$$

$$d_{6q}(\beta, \tau) = \frac{\ln(S_0/K) + (r - q - \frac{1}{2}\sigma^2) \cdot \tau + \sigma\beta}{\sigma \sqrt{\frac{\tau}{2}}}, \quad (3.80)$$

$$\begin{aligned}
\Lambda_{1q}(\tau) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi\tau}} e^{-\frac{\beta^2}{\tau}} e^{-\frac{1}{4}\sigma^2\tau + \sigma\beta} \Phi(d_{5q}(\beta, \tau)) d\beta \\
&= \frac{1}{\pi\sqrt{2\tau}} \int_{-\infty}^{\infty} \int_{-d_{5q}(\beta, \tau)}^{\infty} e^{-\frac{\beta^2}{\tau} - \frac{1}{4}\sigma^2\tau + \sigma\beta - \frac{z^2}{2}} dz d\beta,
\end{aligned} \tag{3.81}$$

$$\begin{aligned}
\Lambda_{2q}(\tau) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi\tau}} e^{-\frac{\beta^2}{\tau}} \Phi(d_{6q}(\beta, \tau)) d\beta \\
&= \frac{1}{\pi\sqrt{2\tau}} \int_{-\infty}^{\infty} \int_{-d_{6q}(\beta, \tau)}^{\infty} e^{-\frac{\beta^2}{\tau} - \frac{w^2}{2}} dw d\beta.
\end{aligned} \tag{3.82}$$

Therefore, we will use this formula in simulations section to compare the price of Sukūk call option with continuous Ijārah  $q$  at time  $t$  with European call option price with continuous dividend determined by the Black-Scholes formula and American call option approximation price.

## 3.4 European Call Option Price

**Definition 3.4.1.** A *European call option* confers its owner the right but not obligation to buy one share of an asset at expiry time  $T$  for the strike price  $K$  [48].

### 3.4.1 Black-Scholes Model

The Black-Scholes formula calculates the price of European call options when the assets are modelled by geometric Brownian motion  $S_t$ , giving an option price of

$$C_E(t) = S_0 \Phi(d_7(\tau)) - K e^{-r\tau} \Phi(d_8(\tau)), \tag{3.83}$$

where

$$\begin{aligned}
d_7 &= \frac{\ln(S_0/K) + (r + \frac{1}{2}\sigma^2) \cdot \tau}{\sigma\sqrt{\tau}} \quad \text{and} \\
d_8(\tau) &= d_7(\tau) - \sigma\sqrt{\tau}, \quad \tau = T - t,
\end{aligned} \tag{3.84}$$

and  $S_0$  is the initial underlying asset price,  $K$  is the strike price,  $r$  is the risk-free interest rate, and  $\sigma$  is the volatility of the return of the underlying asset (cf. [49]).

Similarly, with the value of the European call option with an underlying asset paying continuous dividend  $q$  according to the Black-Scholes model is [49]

$$C_E(t) = S_0 e^{-q\tau} \Phi(d_{7d}(\tau)) - K e^{-r\tau} \Phi(d_{8d}(\tau)), \quad (3.85)$$

where

$$d_{7q} = \frac{\ln(S_0/K) + (r - q + \frac{1}{2}\sigma^2) \cdot \tau}{\sigma\sqrt{\tau}} \quad \text{and} \\ d_{8q}(\tau) = d_{7q}(\tau) - \sigma\sqrt{\tau}, \quad \tau = T - t. \quad (3.86)$$

### 3.5 American Call Option Price Approximation

**Definition 3.5.1.** An *American call option* confers its owner the right but not obligation to buy one share of an asset any time before expiry  $T$  for the strike price  $K$  [48].

According to [17], the right of early exercise makes the American call option at least as valuable as its European counterpart, and is usually worth more. The amount by which the American option's value exceeds the European option is 0 on non-dividend paying assets (Corollary 8.5.3 in [49]). However, in the dividend-paying case, an American option is only optimal at a given time until expiry,  $\tau = T - t$  when the underlying asset price rises above or falls below a critical asset value  $S^*(\tau)$ , the optimal exercise price. The set of critical values constitutes the optimal exercise boundary.

According to [17],

“The early exercise premium can be expressed in terms of the exercise bound-

ary in a stochastic integral. A detailed explanation is given in [32] and [57]. The direct solution of the stochastic integral equation is in many cases unmanageable, so that several analytic approximation methods for the valuation of American options and the associated optimal exercise boundaries have been developed...”

“...The price of the American option is computed by using the popular quadratic approximation method, which was first proposed by [34] for non-dividend paying stock options and later extended to commodity options by [11]. This class of approximation methods involves the reduction of the Black-Scholes partial differential equation to an ordinary one.” [17]

Therefore, an American call option on a dividend-paying asset  $q$  is [17]

$$C_A(S_t, t) = \begin{cases} C_E(S_t, t) + A_2 \left(\frac{S_t}{S^*}\right)^{\gamma_2} & : \text{if } S_t < S^* \\ S - K & : \text{if } S_t \geq S^* \end{cases} \quad (3.87)$$

where  $S^*$  is the critical price of the underlying asset which the option should be exercised. It is estimated by solving the following equation [17]

$$S^* - K = C_E(S_t^*, t) + [1 - e^{-d\tau} \Phi[d_{1q}(S^*)]] \frac{S^*}{\gamma_2}, \quad (3.88)$$

where

$$\gamma_2 = \frac{1}{2} \left[ -\frac{2(r - q) - \sigma^2}{\sigma^2} + \sqrt{\left(\frac{2(r - q) - \sigma^2}{\sigma^2}\right)^2 + \frac{8r}{\sigma^2(1 - e^{-r\tau})}} \right], \quad (3.89)$$

$$A_2 = [1 - e^{-q\tau} \Phi[d_{1q}(S^*)]] \frac{S^*}{\gamma_2}, \quad (3.90)$$

$$d_{1q} = \frac{\ln(S^*/K) + (r - q + \frac{1}{2}\sigma^2) \cdot \tau}{\sigma\sqrt{\tau}}. \quad (3.91)$$

### 3.6 Bounds on Sukūk Call Option Prices

The payoff of a Sukūk call is  $(S_T - K)^+$ , which cannot be negative, which requires a premium must be paid to buy the option. At time  $T$ , then, the gain of the buyer of a Sukūk call is  $(S_T - K)^+ - C_S e^{rT}$ , where  $C_S e^{rT}$  is the risk-free value of the price of the Sukūk option  $C_S$ .

Now we want to establish some upper and lower bounds on the price of Sukūk call option. To avoid arbitrage, each type of option  $C_S$ ,  $C_E$ , and  $C_A$  has an upper bound equals to the initial price of the asset  $S_0$ , so  $C_S < S_0$ .

Otherwise, we could write and sell the option and buy the underlying asset, then sell the underlying asset at exercise time  $T$  for  $\min(S_T, K)$  for an arbitrage profit of  $(C_S - S_0) e^{rT} + \min(S_T, K) > 0$ .

The Sukūk call option also has a lower bound  $S_0 - K e^{-rT} < C_S$ , so

$$S_0 - K e^{-rT} \leq C_S < S_0. \quad (3.92)$$

If  $q$  is the Ijārah, then the Sukūk call option satisfies the following bounds

$$S_0 - K e^{-rT} - q \leq C_S < S_0 - q. \quad (3.93)$$

### 3.7 Comparison of Call Option of Ijārah Sukūk with European and American Call Options

The results of this section will establish price comparisons between American, European, and Sukūk call options. All results below will make use of the Sukūk option

price bounds (3.92).

### 3.7.1 Sukūk Call Option with Ijārah Compared to European Call Option with Dividend at Time $t$

**Lemma 3.7.1.** Suppose the underlying asset to a call option pays Ijārah  $q$  for Sukūk option and dividends  $q$  for European option with the same strike price  $K$  and expiry time  $T$ . The price of the European call option is worth no greater than the price of the Sukūk option,  $C_E(t, q) \leq C_S(t, q) \forall t \in [0, T]$ ,  $q \in \mathbb{R}_+$ .

*Proof.* If the underlying asset pays Ijārah  $q$  for Sukūk and Ijārah  $q$  is equal dividends  $q$ , then  $C_S(t, q) \geq S_0 - Ke^{-rT} - q$ , where  $q > 0$ . As we know, the Sukūk option gives more rights to exercise than the European option, which means that the Sukūk option is worth no less than the European option before mid-term. So, if  $t < \frac{T}{2}$ , then  $C_E(t, q) \leq C_S(t, q)$ . However, if Sukūk option is not exercised at mid-term, then the payoff of Sukūk is the same as that of European option and the holders of the two options have the same rights. Hence, the price of Sukūk option after mid-term will be the same as that of the European option. It means, if  $t \geq \frac{T}{2}$ , then  $C_E(t, q) = C_S(t, q)$ . Therefore,  $C_E(t, q) \leq C_S(t, q) \forall t \in [0, T]$ ,  $q \in \mathbb{R}_+$ .

□

**Theorem 3.7.2.** The prices of Sukūk call options when underlying asset pays Ijārah  $q$  and the European option pays dividend  $q$  and Ijārah  $q$  is equal dividends  $q$  satisfy

$$\max \{0, S_0 - q - Ke^{-rT}\} \leq C_E(t, q) \leq C_S(t, q) < S_0 - q, \quad (3.94)$$

*Proof.* If  $C_S \geq S_0 - q$ , then there would be immediate arbitrage permitted. This implies we must have  $C_S < S_0 - q$ . By Lemma 3.7.1,  $C_E \leq C_S$ , and the first inequality

holds because  $C_E$  cannot be negative or worth less than the initial asset price  $S_0$  minus the dividend and discounted strike price [15]. So we have  $\max \{0, S_0 - q - Ke^{-rT}\} \leq C_E(t, q) \leq C_S(t, q) < S_0 - q$ .  $\square$

### 3.7.2 Sukūk Call Option with Ijārah Compared to American Call Option with Dividend at Time t

**Lemma 3.7.3.** Suppose the underlying asset to a call option pays Ijārah  $q$  for Sukūk option and dividends  $q$  for American option with the same strike price  $K$  and expiry time  $T$ . The price of the Sukūk call option is no greater than the price of the American option,  $C_S(t, q) \leq C_A(t, q) \forall t \in [0, T], q \in \mathbb{R}_+$ .

*Proof.* If the underlying asset pays Ijārah  $q$  for Sukūk and Ijārah  $q$  is equal dividends  $q$ , then  $C_S(t, q) < S_0 - q$ , where  $q > 0$ . Also, if the underlying asset pays dividend  $q$  for American option, then  $C_A(t, q) < S_0$ , where  $q > 0$ . As we know, the American option gives more rights to exercise than the Sukūk option. It means that the Sukūk option worths no greater than the price of the American option. Therefore,  $C_S(t, q) \leq C_A(t, q)$ ,  $\forall t \in [0, T], q \in \mathbb{R}_+$ .  $\square$

### 3.7.3 Sukūk Call Option with Ijārah Compared to European and American Call Option with Dividend at Time t

**Theorem 3.7.4.** The prices of Sukūk, European and American call options on underlying asset that pay dividends  $q$  for European and American and Ijārah  $q$  for Sukūk and Ijārah  $q$  is equal dividends  $q$  satisfy  $C_E(t, q) \leq C_S(t, q) \leq C_A(t, q)$ , each with the same strike price  $K$  and expiry time  $T$ .

*Proof.* We proved  $C_E(t, q) \leq C_S(t, q)$  for options with underlying asset pays dividend for European and Ijārah for Sukūk in Lemma 3.7.1. Similarly, we proved  $C_S(t, q) \leq C_A(t, q)$  in Lemma 3.7.3. Combining these we have  $C_E(t, q) \leq C_S(t, q) \leq C_A(t, q) \forall t \in [0, T]$ ,  $q \in \mathbb{R}_+$ .  $\square$

## 3.8 Time Value of Options

In the subsequent parts of this thesis, we adopt the following terminologies. We say that at time  $t$  a call option of Sukūk with strike price  $K$  is

- “*in the money*” if  $S_t > K$
- “*at the money*” if  $S_t = K$
- “*out of the money*” if  $S_t < K$

where each refers to whether or not there will be a positive payoff if the option were exercised immediately. Depending on the type of call option, of course, it may not be permitted to exercise immediately [44].

The terms have greater importance for American or Sukūk than European call options as they offer more rights to be exercised [15]. Even if a European call option is currently in the money, it may no longer be so on the exercise date, when the payoff may well turn out to be zero. A European option in the money is equivalent to a promising asset, while an American or Sukūk call option in the money offer immediate payoff or may offer payoff at mid-term, respectively.

**Definition 3.8.1.** The *intrinsic value* of a call option with strike price  $K$  at time  $t \leq T$  is  $(S_t - K)^+$  [15].



Clearly only options “in the money” have non-zero intrinsic value, and an option price at expiry  $T$  is exactly the same as the intrinsic value, but option prices before expiry may vary. The price of Sukūk option prior to expiry may be greater than the intrinsic value because of the possibility of future gains. The price of a European option prior to the exercise time may be greater or smaller than the intrinsic value.

**Definition 3.8.2.** The *time value* of Sukūk call option is the difference between the price of the option and its intrinsic value, i.e.

$$C_S(t) - (S_t - K)^+ \quad (3.95)$$

The price of a Sukūk call option “in the money” must be non-negative, as the option may be exercised early at mid-term or at maturity. Typically, the price is higher than the intrinsic value due to the chance of future gains.

The time value of a European call option as a function of  $S$  can never be negative, and for large values of  $S$  it exceeds the difference  $S - Ke^{-rt}$  since we know  $C_E(S) \geq S - Ke^{-rT}$  [15].

For Sukūk option we can exercise at mid-term or maturity to realize the payoff, while for European option we have to wait until the exercise time. The risk that the asset price will rise above  $K$  in the meantime may be considerable, which reduces the value of the option.

## 3.9 Simulations and Discussions

### 3.9.1 Sukūk vs. European Call Option Price at the Beginning Time

The Sukūk call option price at the beginning time was found by (3.66). So, we would like to make simulations to observe the behavior of the call option price of Sukūk with European call option, priced by the Black-Scholes model (3.83). Therefore, in this step we want to see the change in the call option price with respect to one of the parameters  $r$ ,  $K$ ,  $\sigma$  and  $T$ , keeping the other three parameters fixed. Recall that the Sukūk can be exercised at  $T/2$  for any fixed  $T$ .

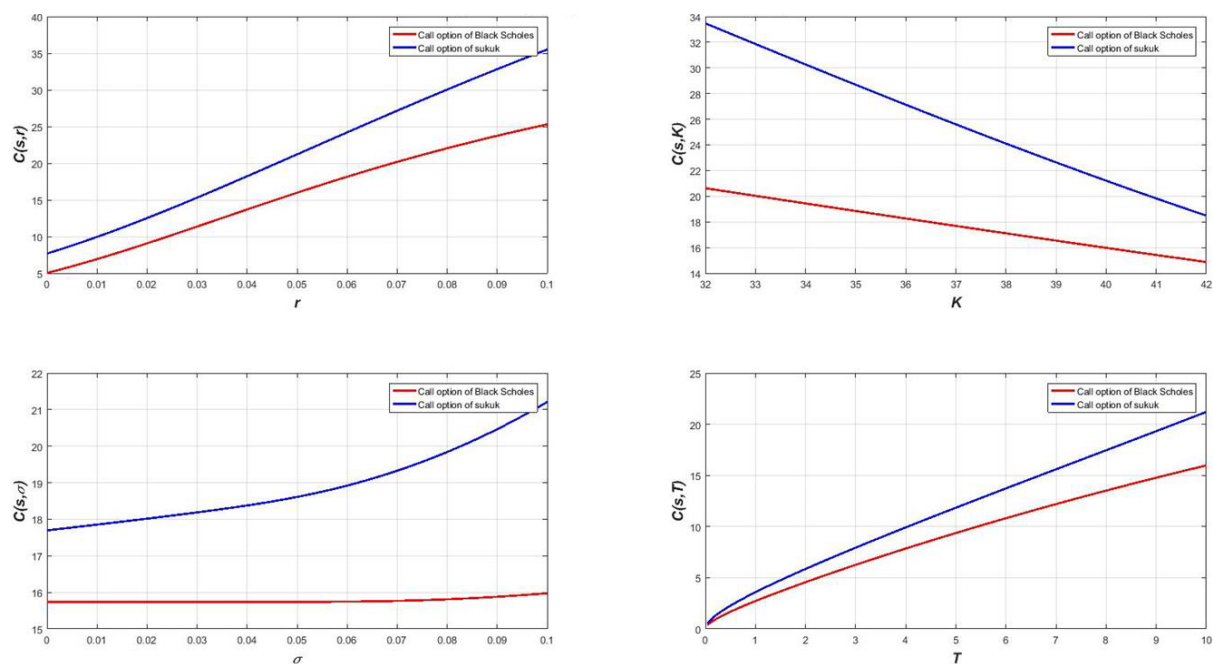


Figure 3.1: Comparison of European and Sukūk call option prices with three parameters fixed and one changing along the horizontal axis. The parameters are the risk-free Ijārah rate  $r$ , the strike price  $K$ , the volatility  $\sigma$ , and the expiry time  $T$

From Figure 3.1, we see that the Sukūk call option price is higher than that of European call option,  $C_E(t = 0) \leq C_S(t = 0)$ , as we expected because the Sukūk option gives more rights to exercise than the European option.

### 3.9.2 Sukūk, European, and American Call Option Price at Time $t$

Using Equation (3.76), we plotted the call option of Sukūk for lifespan and compared it to American and European options.

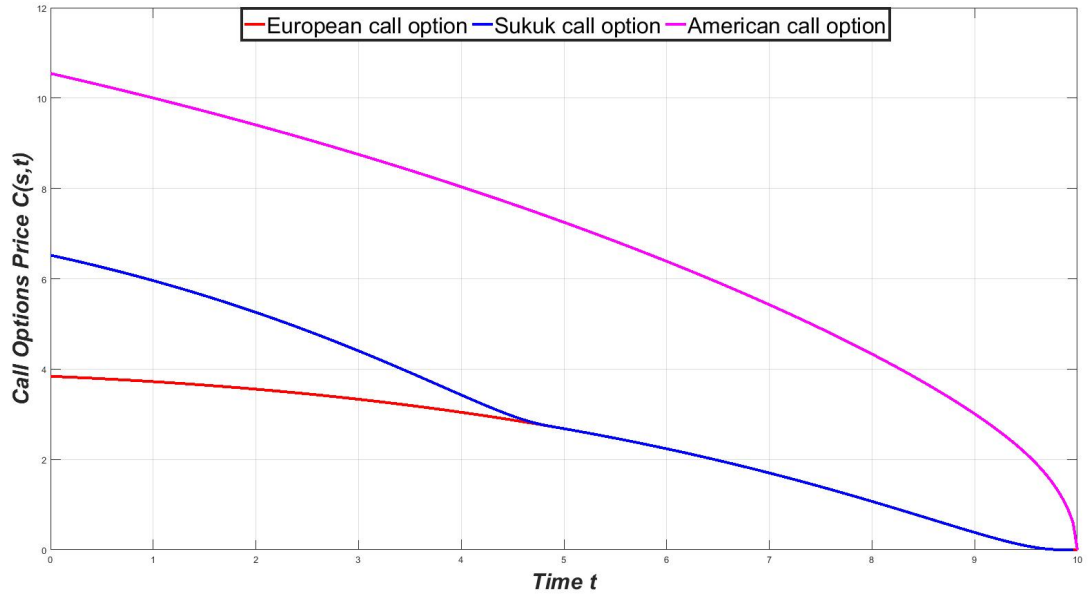


Figure 3.2: European and American call option price with dividend compared to Sukūk call option with Ijārah at time  $t$

Figure 3.2 shows that the Sukūk call option price is greater than the price of the European call option if  $t < \frac{T}{2}$  and it is the same as the European call option price if  $t \geq \frac{T}{2}$  as we have proved in Lemma 3.7.1. Also, the American call option price is greater than the price of the Sukūk and European call options because the American option gives more rights to exercise than Sukūk and European option as we have proved in Lemma 3.7.4. Thus,  $C_E(t, q) \leq C_S(t, q) \leq C_A(t, q)$ .  $\forall t \in [0, T]$ ,  $q \in \mathbb{R}_+$ .

Hence, from the above simulations and discussion we can say that all the numerical simulations support analytical results, theorems, and lemmas that we have proved in this chapter.

# Chapter 4

## Mathematical Modelling of Mid-term Put Option Price of Ijārah Sukūk

### 4.1 Overview

Option derivatives of Sukūk market currently is exercising only at expiry as the European option [18]. However, mid-term exercising of the put option is possible in Ijārah Sukūk [53, 35]. We introduce the mathematical model of a mid-term exercise option in this chapter to demonstrate the pricing possibilities of this idea.

We establish a model along the concept of Ijārah Sukūk, each of them represents ownership of well-defined existing and known assets tied up to a lease contract, Ijārah of which is the return payable to Sukūk holders. On the other hand, buyer of put option has a right but not obligation to sell the underlying asset at mid-term time  $T/2$  and at the expiry time  $T$  if not exercised at mid-term.

## 4.2 Modelling an Ijārah Sukūk Put Option Price

In this section we derive the pricing formula for the mid-term put option of Ijārah Sukūk. Similar to the case of Sukūk call option in Chapter 3, we need first to obtain an appropriate condition which takes care of the choice of exercising at the mid-term. The option should be exercised at mid-term if the payoff at mid-term is more than the expected value of the payoff at expiry discounted at mid-term plus the Ijārah payments that already received till mid-term. This gives us the required condition to exercise at mid-term which can be written as a measurable set. As in the case of Sukūk call option, it can be simplified further by computing the expected value as an integral. Using this condition, the price of the option is then derived by computing the expected value of the discounted payoff under the risk- neutral measure.

### 4.2.1 Statement of the Problem

European put options are exercised only on maturity. However mid-term exercising of the put option is possible in Ijārah Sukūk. Therefore, we will study how the valuation formula changes under such circumstances using both analytical derivation and numerical simulation, and introduce the mathematical model of a mid-term exercise option in this chapter to demonstrate the pricing possibilities of this idea. It means that we want to see the possibility of adjusting the time factor to accommodate this scenario. Moreover, we will compare the price of Sukūk put option with the prices of the corresponding European and American put options for life-span.

**Definition 4.2.1.** The buyer of the *put option* has the right but not the obligation to use the option to sell the underling assets at mid-term of maturity,  $T/2$  or at expiry  $T$ .

### 4.2.2 Early Exercise

- For Sukūk put option, there is some chance that Sukūk option will be exercised early at mid-term  $\frac{T}{2}$
- For European put option, the exercise is only at maturity  $T$ .
- For American put option, the exercise can be carried out at any time over the life-span.

### 4.2.3 Assumptions and Notation

We use the same assumptions and notations as mentioned in Section 3.3.3.

### 4.2.4 Condition for the Buyer of the Put Option to Exercise it at Mid-Term

The important question and mathematical challenge is how the buyer of the put option exercises the option and what is the optimal condition to exercise at mid-term instead of expiry time  $T$ . It is assumed that the buyer of the put option will do so only if he can make more profit or suffer less loss than he does when exercising the option at the expiry time  $T$ , i.e. if the buyer of the put option exercises the option at mid-term, then the following condition should be satisfied:

$$\left(K - S_{t_{\frac{n}{2}}}\right)^+ \geq \mathbb{E} \left[ (K - S_T)^+ e^{-\frac{rT}{2}} \middle| \mathcal{F}_{t_{\frac{n}{2}}} \right] + \sum_{i=1}^{\frac{n}{2}} q e^{-\frac{rTi}{n}}, \quad (4.1)$$

where  $q$  is the periodic distribution amount (the Ijārah payment). Let  $A \in \mathcal{F}$  be the set where this is true, i.e.

$$A = \left\{ \omega \in \Omega : \left( K - S_{t_{\frac{n}{2}}} \right)^+ \geq \mathbb{E} \left[ (K - S_T)^+ e^{-\frac{rT}{2}} \middle| \mathcal{F}_{t_{\frac{n}{2}}} \right] + \sum_{i=1}^{\frac{n}{2}} q e^{-\frac{rTi}{n}} \right\}. \quad (4.2)$$

Since the left-hand side is the payoff of the put option at mid-term while the right-hand side is the discounted expected payoff of the put option based on the information we have at mid-term. Therefore, this is the condition that when the buyer of the put option should exercise it.

As known in above assumptions, the price of put option is the discounted expected payoff under measure  $\mathbb{Q}$ . Therefore, the price of the put option of Sukūk based on the above condition can be decomposed into

$$P_S = e^{-\frac{rT}{2}} \mathbb{E}^{\mathbb{Q}} \left[ \left( K - S_{t_{\frac{n}{2}}} \right)^+ \middle| A \right] + e^{-rT} \mathbb{E}^{\mathbb{Q}} \left[ (K - S_T)^+ \middle| A^c \right]. \quad (4.3)$$

#### 4.2.5 Simplification of the Condition

By following the analogous computations as in Section 3.3.5, we can simplify the condition in (4.1) to obtain

$$\left( K - S_{t_{\frac{n}{2}}} \right)^+ \geq K e^{-\frac{1}{2}rT} \Phi(-d_2(T)) - S_{t_{\frac{n}{2}}} e^{\frac{T}{2}\mu - \frac{1}{2}rT} \Phi(-d_1(T)) + \sum_{i=1}^{\frac{n}{2}} q e^{-\frac{rTi}{n}}. \quad (4.4)$$

## 4.2.6 Derivations of Put Options Price Formulas

We can derive the price of the put option (4.3) by following the similar arguments as used for the call option in Section 3.3.6. Notice first that (4.4) is equivalent to

$$\left\{ 0 \leq K e^{-\frac{1}{2}rT} \Phi(-d_2(T)) \leq S_{t_{\frac{n}{2}}} e^{\frac{T}{2}\mu - \frac{1}{2}rT} \Phi(-d_1(T)) - \sum_{i=1}^{\frac{n}{2}} q e^{-\frac{rTi}{n}} \right\} \quad (4.5)$$

on  $\{K \leq S_{t_{\frac{n}{2}}}\}$  and

$$\left\{ K - S_{t_{\frac{n}{2}}} \geq K e^{-\frac{1}{2}rT} \Phi(-d_2(T)) - S_{t_{\frac{n}{2}}} e^{\frac{T}{2}\mu - \frac{1}{2}rT} \Phi(-d_1(T)) + \sum_{i=1}^{\frac{n}{2}} q e^{-\frac{rTi}{n}} > 0 \right\}, \quad (4.6)$$

on  $\{K > S_{t_{\frac{n}{2}}}\}$ .

Setting  $K$  such that

$$K \leq \frac{K - K e^{-\frac{1}{2}rT} \Phi(-d_2(T)) - \sum_{i=1}^{\frac{n}{2}} q e^{-\frac{rTi}{n}}}{1 - e^{\frac{T}{2}\mu - \frac{1}{2}rT} \Phi(-d_1(T))}, \quad (4.7)$$

and  $1 - e^{\frac{T}{2}\mu - \frac{1}{2}rT} \Phi(-d_1(T)) > 0$ , we can verify that the condition (4.4) holds. Using analogous computations as in Section 3.3.6 the first part of the put option price (4.3) becomes

$$e^{-\frac{rT}{2}} \mathbb{E}^{\mathbb{Q}} \left[ \left( K - S_{t_{\frac{n}{2}}} \right)^+ \middle| A \right] = K e^{-r\frac{T}{2}} \Phi \left( -d_4 \left( \frac{T}{2} \right) \right) - S_0 \Phi \left( -d_3 \left( \frac{T}{2} \right) \right), \quad (4.8)$$

where

$$d_3 \left( \frac{T}{2} \right) = \frac{\ln(S_0/K) + (r + \frac{1}{2}\sigma^2) \cdot \frac{T}{2}}{\sigma \sqrt{\frac{T}{2}}}, \quad (4.9)$$

$$d_4 \left( \frac{T}{2} \right) = \frac{\ln(S_0/K) + (r - \frac{1}{2}\sigma^2) \cdot \frac{T}{2}}{\sigma \sqrt{\frac{T}{2}}}. \quad (4.10)$$



while the second part of (4.3) can be simplified as

$$e^{-rT} \mathbb{E}^{\mathbb{Q}} [(S_T - K)^+ | A^c] = K e^{-rT} \mathbf{\Lambda}_2(T) - S_0 \mathbf{\Lambda}_1(T), \quad (4.11)$$

where

$$\begin{aligned} \mathbf{\Lambda}_1(T) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi T}} e^{-\frac{\beta^2}{T}} e^{-\frac{1}{4}\sigma^2 T + \sigma\beta} \Phi(-d_5(\beta, T)) d\beta \\ &= \frac{1}{\pi\sqrt{2T}} \int_{-\infty}^{\infty} \int_{-\infty}^{d_5(\beta, T)} e^{-\frac{\beta^2}{T} - \frac{1}{4}\sigma^2 T + \sigma\beta - \frac{z^2}{2}} dz d\beta, \end{aligned} \quad (4.12)$$

$$\begin{aligned} \mathbf{\Lambda}_2(T) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi T}} e^{-\frac{\beta^2}{T}} \Phi(-d_6(\beta, T)) d\beta \\ &= \frac{1}{\pi\sqrt{2T}} \int_{-\infty}^{\infty} \int_{-\infty}^{d_6(\beta, T)} e^{-\frac{\beta^2}{T} - \frac{w^2}{2}} dw d\beta, \end{aligned} \quad (4.13)$$

$$d_5(\beta, T) = \frac{\ln(S_0/K) + rT + \sigma\beta}{\sigma\sqrt{\frac{T}{2}}}, \quad (4.14)$$

$$d_6(\beta, T) = \frac{\ln(S_0/K) + (r - \frac{1}{2}\sigma^2)T + \sigma\beta}{\sigma\sqrt{\frac{T}{2}}}. \quad (4.15)$$

Combining (4.8) and (4.11), we find a formula for the price of the Sukūk put option,

$$P_S(0) = \left[ K e^{-r\frac{T}{2}} \Phi\left(-d_4\left(\frac{T}{2}\right)\right) - S_0 \Phi\left(-d_3\left(\frac{T}{2}\right)\right) \right] + [K e^{-rT} \mathbf{\Lambda}_2(T) - S_0 \mathbf{\Lambda}_1(T)]. \quad (4.16)$$

#### 4.2.7 Mid-term Put Option of Sukūk at Time $t$

Analogous to Chapter 3, we can derive the formula for price of mid-term put option of Sukūk at any time  $t$  given by

$$P_S(t) = \mathbb{1}_{\{t \leq \frac{T}{2}\}} [K e^{-r\tau_s} \Phi(-d_4(\tau_s)) - S_0 \Phi(-d_3(\tau_s))] + [K e^{-r\tau} \mathbf{\Lambda}_2(\tau) - S_0 \mathbf{\Lambda}_1(\tau)], \quad (4.17)$$

where,  $d_3(\tau_s)$ ,  $d_4(\tau_s)$ ,  $d_5(\tau_s)$ ,  $d_6(\tau_s)$  are given by (3.68), (3.69), (3.70), (3.71) respectively and

$$\begin{aligned}\Lambda_1(\tau) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi\tau}} e^{-\frac{\beta^2}{\tau}} e^{-\frac{1}{4}\sigma^2\tau+\sigma\beta} \Phi(-d_5(\beta, \tau)) d\beta \\ &= \frac{1}{\pi\sqrt{2\tau}} \int_{-\infty}^{\infty} \int_{-\infty}^{d_5(\beta, \tau)} e^{-\frac{\beta^2}{\tau}-\frac{1}{4}\sigma^2\tau+\sigma\beta-\frac{z^2}{2}} dz d\beta,\end{aligned}\tag{4.18}$$

$$\begin{aligned}\Lambda_2(\tau) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi\tau}} e^{-\frac{\beta^2}{\tau}} \Phi(-d_6(\beta, \tau)) d\beta \\ &= \frac{1}{\pi\sqrt{2\tau}} \int_{-\infty}^{\infty} \int_{-\infty}^{d_6(\beta, \tau)} e^{-\frac{\beta^2}{\tau}-\frac{w^2}{2}} dw d\beta.\end{aligned}\tag{4.19}$$

#### 4.2.8 Mid-term Put Option of Sukūk with Continuous Ijārah at Time $t$

Similar to Section 3.3.8, we can obtain Sukūk put option price with continuous Ijārah  $q$  given by

$$P_S(t) = \mathbb{1}_{\{t \leq \frac{T}{2}\}} [K e^{-r\tau_s} \Phi(-d_{4q}(\tau_s)) - S_0 e^{-q\tau_s} \Phi(-d_{3q}(\tau_s))] + [K e^{-r\tau} \Lambda_{2q}(\tau) - S_0 e^{-q\tau} \Lambda_{1q}(\tau)],\tag{4.20}$$

where,  $d_{3q}(\tau_s)$ ,  $d_{4q}(\tau_s)$ ,  $d_{5q}(\tau_s)$ ,  $d_{6q}(\tau_s)$  are given by (3.77), (3.78), (3.79), (3.80) respectively and

$$\begin{aligned}\Lambda_{1q}(\tau) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi\tau}} e^{-\frac{\beta^2}{\tau}} e^{-\frac{1}{4}\sigma^2\tau+\sigma\beta} \Phi(-d_{5q}(\beta, \tau)) d\beta \\ &= \frac{1}{\pi\sqrt{2\tau}} \int_{-\infty}^{\infty} \int_{-\infty}^{d_{5q}(\beta, \tau)} e^{-\frac{\beta^2}{\tau}-\frac{1}{4}\sigma^2\tau+\sigma\beta-\frac{z^2}{2}} dz d\beta,\end{aligned}\tag{4.21}$$

$$\begin{aligned}\Lambda_{2q}(\tau) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi\tau}} e^{-\frac{\beta^2}{\tau}} \Phi(-d_{6q}(\beta, \tau)) d\beta \\ &= \frac{1}{\pi\sqrt{2\tau}} \int_{-\infty}^{\infty} \int_{-\infty}^{d_{6q}(\beta, \tau)} e^{-\frac{\beta^2}{\tau}-\frac{w^2}{2}} dw d\beta.\end{aligned}\tag{4.22}$$

## 4.3 European Put Option Price

**Definition 4.3.1.** A *European put option* confers its owner the right but not obligation to sell a share of an asset at expiry time  $T$  for the strike price  $K$  [48].

### 4.3.1 Black-Scholes Model

The Black-Scholes formula calculates the price of European put options when the assets are modelled by geometric Brownian motion  $S_t$ , giving an option price of

$$P_E(t) = Ke^{-r\tau}\Phi(-d_8(\tau)) - S_0\Phi(-d_7(\tau)), \quad (4.23)$$

where

$$d_7 = \frac{\ln(S_0/K) + (r + \frac{1}{2}\sigma^2) \cdot \tau}{\sigma\sqrt{\tau}} \quad \text{and} \\ d_8(\tau) = d_7(\tau) - \sigma\sqrt{\tau}, \quad \tau = T - t, \quad (4.24)$$

here  $S_0$  is the initial underlying asset price,  $K$  is the strike price,  $r$  is the risk-free interest rate, and  $\sigma$  is the volatility of the return of the underlying asset [49].

Similarly, with the value of the European put option with an underlying asset paying continuous dividend  $q$  according to the Black-Scholes model is (cf. [49])

$$P_E(t) = Ke^{-r\tau}\Phi(-d_{8q}(\tau)) - S_0e^{-q\tau}\Phi(-d_{7q}(\tau)), \quad (4.25)$$

where

$$d_{7q} = \frac{\ln(S_0/K) + (r - q + \frac{1}{2}\sigma^2) \cdot \tau}{\sigma\sqrt{\tau}} \quad \text{and} \\ d_{8q}(\tau) = d_{7q}(\tau) - \sigma\sqrt{\tau}, \quad \tau = T - t. \quad (4.26)$$

## 4.4 American Put Option Price Approximation

**Definition 4.4.1.** An *American put option* confers its owner the right but not obligation to sell a share of an asset any time before expiry  $T$  for the strike price  $K$  [48].

An American put option on a dividend-paying asset  $q$  is [17]

$$P_A(S_t, t) = \begin{cases} P_E(S_t, t) + A_1 \left(\frac{S_t}{S^{**}}\right)^{\gamma_1} & : \text{if } S_t > S^{**} \\ K - S & : \text{if } S_t \leq S^{**} \end{cases} \quad (4.27)$$

where  $S^{**}$  is the critical price of the underlying asset which the option should be exercised.

It is estimated by solving the following equation [17]

$$K - S^{**} = P_E(S_t^{**}, t) - [1 - e^{-q\tau} \Phi[-d_{1q}(S^{**})]] \frac{S^{**}}{\gamma_1}, \quad (4.28)$$

where

$$\gamma_1 = \frac{1}{2} \left[ -\frac{2(r - q) - \sigma^2}{\sigma^2} - \sqrt{\left(\frac{2(r - q) - \sigma^2}{\sigma^2}\right)^2 + \frac{8r}{\sigma^2(1 - e^{-r\tau})}} \right], \quad (4.29)$$

$$A_1 = -[1 - e^{-q\tau} \Phi[-d_{1q}(S^{**})]] \frac{S^{**}}{\gamma_1}, \quad (4.30)$$

$$d_{1q} = \frac{\ln(S^{**}/K) + (r - q + \frac{1}{2}\sigma^2) \cdot \tau}{\sigma\sqrt{\tau}}. \quad (4.31)$$

## 4.5 Bounds on Sukūk Put Option Prices

The payoff of a Sukūk put is  $(K - S_T)^+$ , which cannot be negative, which requires a premium must be paid to sell the option. At time  $T$ , then, the gain of the buyer of option is  $P_S e^{rT} - (K - S_T)^+$ , where  $P_S e^{rT}$  is the risk-free value of the price of the Sukūk option  $P_S$ .

Now we want to establish some upper and lower bounds on the price of Sukūk put option without dividend. To avoid arbitrage, each type of option  $P_S$ ,  $P_E$ , and  $P_A$  has an upper bound equals to  $Ke^{-rT}$ .

The Sukūk put option also has a lower bound  $Ke^{-rT} - S_0 < P_S$ , so

$$Ke^{-rT} - S_0 \leq P_S < Ke^{-rT}. \quad (4.32)$$

If  $q$  is the Ijārah, then the Sukūk put option satisfies the following bounds

$$Ke^{-rT} - S_0 - q \leq P_S < Ke^{-rT} - q. \quad (4.33)$$

## 4.6 Comparison of Put Option of Ijārah Sukūk with European and American Put Options

The results of this section will establish price comparisons between American, European, and Sukūk put options. All results below will make use of the Sukūk option price bounds (4.32).

### 4.6.1 Sukūk Put Option with Ijārah Compared to European Put option with Dividend at Time $t$

**Lemma 4.6.1.** Suppose the underlying asset to a put option pays Ijārah  $q$  for Sukūk option and dividends  $q$  for European option and Ijārah  $q$  is equal dividends  $q$  with the same strike price  $K$  and expiry time  $T$ . The price of the European put option is no greater than the price of the Sukūk option,  $P_E(t, q) \leq P_S(t, q) \forall t \in [0, T]$ ,  $q \in \mathbb{R}_+$ .

*Proof.* If the underlying asset pays Ijārah  $q$  for Sukūk, then  $P_S(t, q) \geq Ke^{-rT} + q - S_0$ , where  $q > 0$ . As we know, the Sukūk option gives more rights to exercise than the European option, which means that the Sukūk option is worth no less than the European option before mid-term. So, if  $t < \frac{T}{2}$ , then  $P_E(t, q) \leq P_S(t, q)$ . However, if Sukūk option is not exercised at mid-term, then the payoff of the two options are the same and the holders of these options have the same rights. Hence, the price will be the same European option after mid-term. It means, if  $t \geq \frac{T}{2}$ , then  $P_E(t, q) = P_S(t, q)$ . Therefore,  $P_E(t, q) \leq P_S(t, q) \forall t \in [0, T], q \in \mathbb{R}_+$ .

□

**Theorem 4.6.2.** The prices of Sukūk put options when underlying asset pays Ijārah  $q$  and the European option pays dividend  $q$  and both Ijārah and dividends are equal satisfy

$$\max \{0, Ke^{-rT} - S_0 + q\} \leq P_E(t, q) \leq P_S(t, q) < Ke^{-rT}, \quad (4.34)$$

*Proof.* If  $P_S \geq Ke^{-rT}$ , there would be an immediate arbitrage permitted. Therefore, we must have  $P_S < Ke^{-rT}$ . By Lemma 4.6.3,  $P_E(t, q) \leq P_S(t, q)$ , and the first inequality holds because  $P_E$  cannot be negative or worth less than discounted strike price [15]. So, we have  $\max \{0, Ke^{-rT} - S_0 + q\} \leq P_E(t, q) \leq P_S(t, q) < Ke^{-rT}$ . □

#### 4.6.2 Sukūk Put Option with Ijārah Compared to American Put option with Dividend at Time $t$

**Lemma 4.6.3.** Suppose the underlying asset to a put option pays Ijārah  $q$  for Sukūk option and dividends  $q$  for American option, and both Ijārah and dividends are equal with the same strike price  $K$  and expiry time  $T$ . The price of the Sukūk put option is no greater than the price of the American option,  $P_S(t) \leq P_A(t) \forall t \in [0, T], q \in \mathbb{R}_+$ .

*Proof.* If the underlying asset pays Ijārah  $q$  for Sukūk, then  $P_S(t, q) \geq Ke^{-rT} - S_0 + q$ , where  $q > 0$ . Also, If the underlying asset pays dividend  $q$  for American, then  $P_A(t, q) \geq Ke^{-rT} - S_0 + q$ , where  $q > 0$ . As we know, the American option gives more rights to exercise than the Sukūk option, which means that the Sukūk option is worth no greater than the price of the American option. Therefore,  $P_S(t, q) \leq P_A(t, q) \forall t \in [0, T]$ ,  $q \in \mathbb{R}_+$ .  $\square$

### 4.6.3 Sukūk Put Option with Ijārah Compared to European and American Put option with Dividend at Time t

**Theorem 4.6.4.** The prices of Sukūk, European and American put options on underlying asset pays dividends  $q$  satisfy  $P_E(t) \leq P_S(t) \leq P_A(t)$ , each with the same strike price  $K$  and expiry time  $T$  and both Ijārah and dividends are equal.

*Proof.* We proved  $P_E(t, q) \leq P_S(t, q)$  for options with underlying asset pays dividend for European and Ijārah for Sukūk in Lemma 4.6.1. Similarly, we proved  $P_S(t, q) \leq P_A(t, q)$  in Lemma 4.6.3. Combining these we have  $P_E(t, q) \leq P_S(t, q) \leq P_A(t, q) \forall t \in [0, T]$ ,  $q \in \mathbb{R}_+$ .  $\square$

## 4.7 Time Value of Options

The following convenient terminology is often used. We say that at time  $t$  a put option of Sukūk with strike price  $K$  is

- “*in the money*” if  $K > S_t$
- “*at the money*” if  $K = S_t$
- “*out of the money*” if  $K < S_t$

where each refers to whether or not there will be a positive payoff if the option were exercised immediately. Depending on the type of put option, of course, it may not be permitted to exercise immediately [44].

The terms have greater importance for American or Sukūk than European put options as they offer more rights to be exercised [15]. Even if a European put option is currently in the money, it may no longer be so on the exercise date, when the payoff may well turn out to be zero. A European option in the money is equivalent to a promising asset, while an American or Sukūk put option in the money offer immediate payoff or may offer payoff at mid-term, respectively.

**Definition 4.7.1.** The *intrinsic value* of a put option with strike price  $K$  at time  $t \leq T$  is  $(K - S_t)^+$  [15].

Clearly only options “in the money” have non-zero intrinsic value, and an option price at expiry  $T$  is exactly as the same the intrinsic value, but option prices before expiry may vary. The price of Sukūk option prior to expiry may be greater than the intrinsic value because of the possibility of future gains. The price of a European option prior to the exercise time may be greater or smaller than the intrinsic value.

**Definition 4.7.2.** The *time value* of Sukūk put option is the difference between the price of the option and its intrinsic value, i.e.

$$P_S(t) - (K - S_t)^+ \tag{4.35}$$

The price of a Sukūk put option “in the money” must be non-negative, as the option may be exercised early at mid-term or at maturity. Typically, the price is higher than the intrinsic value due to the chance of future gains.

The time value of a European put option as a function of  $S$  can never be negative, and for large values of  $S$  it exceeds the difference  $Ke^{-rt} - S$  since we know  $P_E(S) \geq$



$$Ke^{-rT} - S \text{ [15].}$$

For Sukūk option we can exercise at mid-term or maturity to realize the payoff, while for European option we have to wait until the exercise time. The risk that the asset price will rise above  $K$  in the meantime may be considerable, which reduces the value of the option.

## 4.8 Simulations and Discussions

### 4.8.1 Sukūk vs. European Put Option Price at the Beginning Time

The Sukūk put option price at the beginning time was found by (4.16). So, we would like to make simulations to observe the behavior of the put option price of Sukūk with European put option, priced by the Black-Scholes model (4.23). Therefore, in this step we want to see the put option price when we fix the parameters  $r$ ,  $K$ ,  $\sigma$  and  $T$ . Recall that the Sukūk can be exercised at  $T/2$  for any fixed  $T$ .

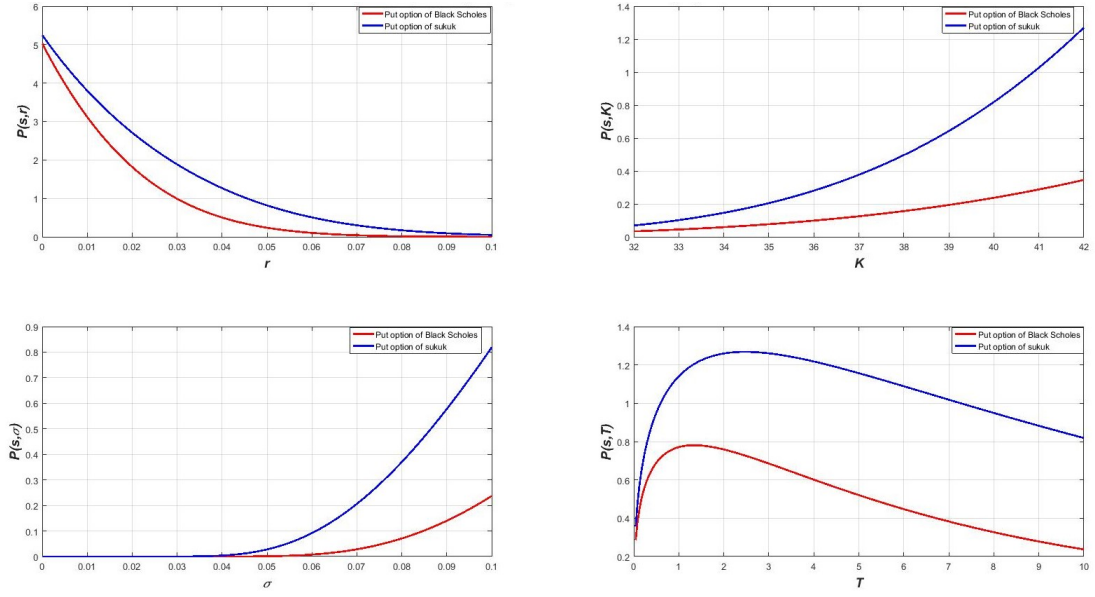


Figure 4.1: Comparison of European and Sukūk put option prices with three parameters fixed and one changing along the horizontal axis. The parameters are the risk-free Ijārah rate  $r$ , the strike price  $K$ , the volatility  $\sigma$ , and the expiry time  $T$

From Figure 4.1, we see that the Sukūk put option price is higher than that of European put option,  $P_E(t = 0) \leq P_S(t = 0)$ , as we expected because the Sukūk option gives more rights to exercise than the European option.

#### 4.8.2 Sukūk, European, and American Put Option Price at Time $t$

Using Equation (4.20), we plotted the put option of Sukūk with Ijārah for lifespan and compared it to American and European options with dividend.

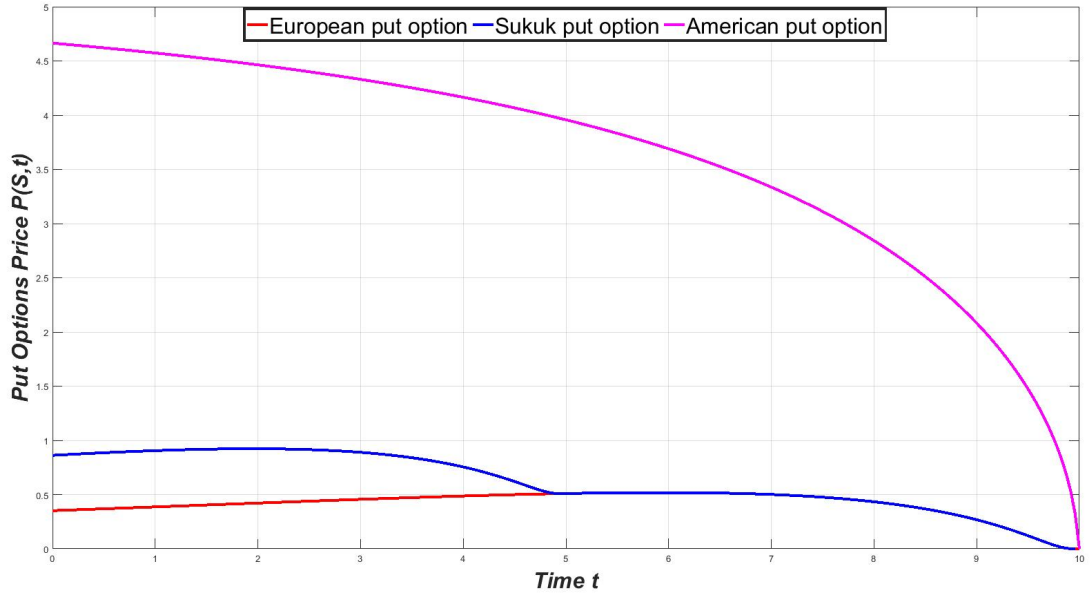


Figure 4.2: European and American put option price with dividend compared to Sukūk put option with Ijārah at time  $t$

Figure 4.2 shows that the price of Sukūk put option price is greater than the price of the European put option if  $t < \frac{T}{2}$  and the same as the European put option price if  $t \geq \frac{T}{2}$  as we have proved in Lemma 4.6.1. Also, the American put option price is greater than the price of the Sukūk and European put option because the American option gives more rights to exercise than Sukūk and European option as we have proved in Lemma 4.6.3 . Thus,  $P_E(t, q) \leq P_S(t, q) \leq P_A(t, q)$ .  $\forall t \in [0, T]$  ,  $q \in \mathbb{R}_+$ .

Hence, from the above simulations and discussion we observed that all the numerical simulations support analytical results, theorems, and lemmas that we have proved in this chapter.

# Chapter 5

## A Mid-term Callable and Puttable Sukūk Price of Ijārah Sukūk

### 5.1 A Mid-term Callable Sukūk

#### 5.1.1 Overview

A callable Sukūk, also called redeemable Sukūk, gives the issuer of Sukūk the right, but not the obligation to redeem the Sukūk from the Sukūk holders at a defined call price prior to its maturity date [10]. In Sukūk, the defined date to redeem the Sukūk is at mid-term of maturity. The feature of a callable Sukūk for the issuer is that it gives flexibility in repaying Ijārah (rent) obligations. For example, a typical callable Sukūk may have a maturity six years but allow the issuer to call the Sukūk at face value after three years at mid-term. After three years has passed, the issuer can look at the Sukūk market conditions and take a decision whether to call the Sukūk or leave it outstanding.

### 5.1.2 Statement of the Problem

As known in the European callable bond, the issuer can redeem the bond at a date before the maturity such as the last coupon date which is a single call date [28]. However, the American callable bond can be redeemed by the issuer at any time before maturity [28]. On the other hand, the callable Sukūk is redeemed by the issuer of Sukūk at mid-term of its life. Therefore, we aim to find the callable Sukūk price based on the mid-term call option of Sukūk that we derived in previous chapter. Then, we will compare the callable Sukūk price with the corresponding European and American callable bond prices during lifespan. Moreover, we want to study the relationship between price of callable Sukūk and the Ijārah rate by the duration and convexity.

### 5.1.3 Early Callable Bond and Sukūk

- For Sukūk callable, there is some chance that the Sukūk will be called early at the mid-term of its life.
- For European callable, the call is only at maturity.
- For American callable, the call can be carried out at any time of its life.

### 5.1.4 Assumptions and Notations

We use the same assumptions and notations as in the previous Chapters 3 and 4. Also, we added these notations for this chapter  $S_C$ ,  $E_C$  and  $A_C$  which are the callable Sukūk, European and American callable bond, respectively.

### 5.1.5 Callable Sukūk Price

A callable Sukūk is a straight Sukūk embedded with mid-term call option of Sukūk [51]. The price of straight Sukūk is defined at par value. If the callable Sukūk price exceeds the exercise price at mid-term, the issuer will exercise the option. Otherwise, the issuer will not use the right to redeem the Sukūk because the non-callable Sukūk price is equal to the callable Sukūk price. Hence, the callable Sukūk price will be lower than the straight Sukūk price and the yield on callable Sukūk is worth more than the yield on straight Sukūk. Therefore, the price of callable Sukūk with Ijārah  $q$  at time  $t$  is

$$\text{Price of callable Sukūk} = \text{Price of straight Sukūk} - \text{Call option price.} \quad (5.1)$$

Using the formula derived in Chapter 3, we have

$$S_C(t) = FV - (\mathbb{1}_{\{t \leq \frac{T}{2}\}} [S_0 e^{-q\tau_s} \Phi(d_{3q}(\tau_s)) - K e^{-r\tau_s} \Phi(d_{4q}(\tau_s))] + [S_0 e^{-q\tau} \Lambda_{1q}(\tau) - K e^{-r\tau} \Lambda_{2q}(\tau)]). \quad (5.2)$$

In this section, we assume that the straight Sukūk price is the face value or par value and find the mid-term call option price as the same as in the Chapter 3. Note that, we will use in this chapter the terminology of Ijārah which corresponds to the coupon in western finance.

Most kinds of Sukūk are worth their par value and the sum of Ijārah until maturity, while the callable Sukūk are worth less because the issuer of Sukūk may redeem them at mid-term (before the maturity date). If rental (Ijārah) rates drops before the Sukūk mid-term, the issuer may exercise the option and refinance it at a lower rate.

### 5.1.6 European Callable Bond Price

The European callable bond is a bond that allows the issuer of the bond to redeem the bond prior to maturity. Therefore, the price of European callable bond is always less than the straight bond price. Also, the yield on European callable bond is worth more than the yield on straight bond [28]. So, the formula of European callable bond price with coupon at time  $t$  is

$$\text{Price of European callable bond} = \text{Price of straight bond} - \text{Call option price}, \quad (5.3)$$

$$E_C(t) = FV - (S_0 e^{-q\tau} \Phi(d_{7q}(\tau)) - K e^{-r\tau} \Phi(d_{8q}(\tau))). \quad (5.4)$$

In this section, we assume that the straight bond price is the face value and find the European call option price as the same as in the Chapter 3 by the Black-Scholes formula.

### 5.1.7 American Callable Bond Price

The American callable bond is a bond that allows the issuer of the bond to redeem the bond at any time before maturity. Therefore, the price of American callable bond is always less than the straight bond price [28]. So, the formula of American callable bond price is

$$\text{Price of American callable bond} = \text{Price of straight bond} - \text{Call option price} \quad (5.5)$$

In this section, we assume that the straight bond price is the face value and find the American call option price as the same as in the Chapter 3 by the approximation methods.

## 5.2 Comparison of Callable Sukūk with European and American Callable Bonds

### 5.2.1 Callable Sukūk with Ijārah Compared to European Callable Bond Price with Coupon at Time t

**Lemma 5.2.1.** Suppose the underlying asset pays Ijārah  $q$  for callable Sukūk and coupon  $q$  for European callable bond with the same strike price  $K$  and expiry time  $T$ . The price of the callable Sukūk is no greater than the price of the European callable bond,  $S_C(t, q) \leq E_C(t, q) \forall t \in [0, T]$  ,  $q \in \mathbb{R}_+$ .

*Proof.* Assume that the straight price or face value is the same and Ijārah  $q$  is equal coupon  $q$ . Also, as proved in Lemma 3.7.1 that the price of Sukūk call option is greater than the price of the European call option if  $t < \frac{T}{2}$  and the same as that of European call option price if  $t \geq \frac{T}{2}$ . Therefore, the difference between price of straight bond and call option price for European callable bond is grater than in Sukūk callable. Thus  $S_C(t, q) \leq E_C(t, q) \forall t \in [0, T]$  ,  $q \in \mathbb{R}_+$ .  $\square$

### 5.2.2 Callable Sukūk with Ijārah Compared to American Callable Bond Price with Coupon at Time t

**Lemma 5.2.2.** Suppose the underlying asset pays Ijārah  $q$  for callable Sukūk and coupon  $q$  for American callable bond with the same strike price  $K$  and expiry time  $T$ . The price of the American callable bond is no greater than the price of callable Sukūk,  $A_C(t, q) \leq S_C(t, q) \forall t \in [0, T]$  ,  $q \in \mathbb{R}_+ \forall t \in [0, T]$  ,  $q \in \mathbb{R}_+$ .

*Proof.* Assume that the straight price or face value is the same and Ijārah  $q$  is equal



coupon  $q$ . Also, as proved in Lemma 3.7.3 that the price of Sukūk call option is no greater than the price of the American call option for all time  $t$ . Therefore, the difference between price of straight bond and call option price for Sukūk callable bond is grater than in American callable. Thus  $A_C(t, q) \leq S_C(t, q) \forall t \in [0, T] , q \in \mathbb{R}_+$ .  $\square$

### 5.2.3 Callable Sukūk with Ijārah Compared to European and American Callable Bond Price with Coupon at Time t

**Theorem 5.2.3.** The prices of callable Sukūk with Ijārah  $q$ , European and American callable bonds with coupon  $q$  with the same strike price  $K$  and expiry time  $T$  satisfy  $A_C(t, q) \leq S_C(t, q) \leq E_C(t, q) \forall t \in [0, T] , q \in \mathbb{R}_+$ .

*Proof.* We proved that  $S_C(t, q) \leq E_C(t, q)$  in Lemma 5.2.1. Similarly, we proved  $A_C(t, q) \leq S_C(t, q)$  in Lemma 5.2.2. Combining these we have  $A_C(t, q) \leq S_C(t, q) \leq E_C(t, q)$ .  $\forall t \in [0, T] , q \in \mathbb{R}_+$ .  $\square$

## 5.3 Simulations and Discussions

In this section, we want to simulate the callable Sukūk price versus the European and American callable bond and see the prices' behavior over lifespan. We assume that the straight Sukūk price is the face value or par value and find the call option price as the same in previous chapters.

### 5.3.1 Callable Sukūk Price with Ijārah Compared to European and American Callable Bond with Coupon at Time $t$

In this step, we plotted the callable Sukūk price over lifespan and compared it to American and European callable bond.

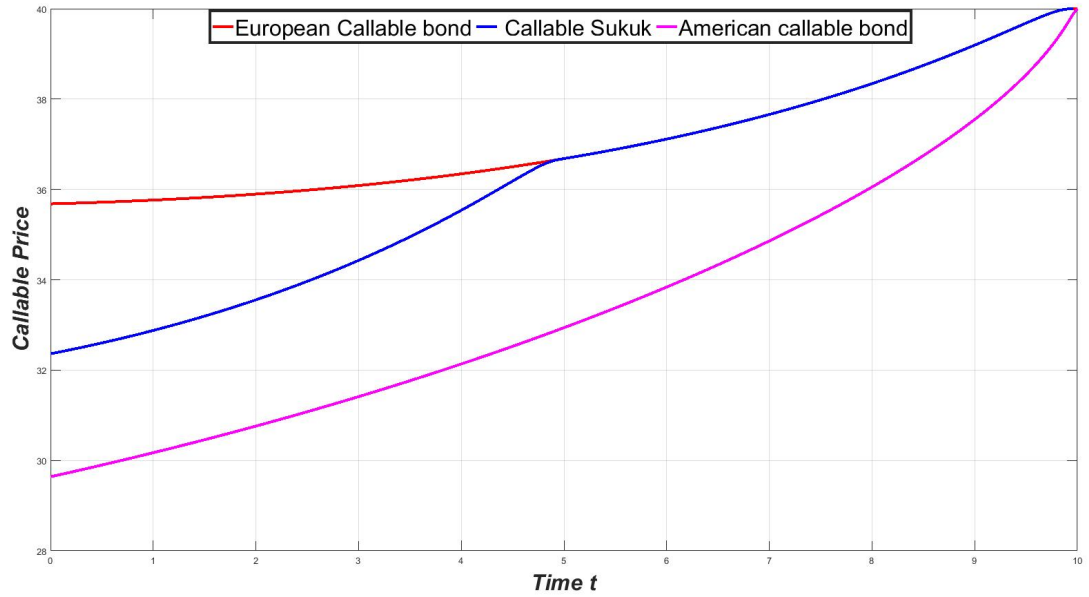


Figure 5.1: European and American callable bond price with coupon compared to callable Sukūk with Ijārah at time  $t$ .

Figure 5.1 shows that the price of Sukūk callable is no greater than the price of the European callable bond if  $t < \frac{T}{2}$  and the same as European callable bond price if  $t \geq \frac{T}{2}$  as we have proved in Lemma 5.2.1. Also, the Sukūk callable price is greater than the price of the American callable bond as we have proved in Lemma 5.2.2. Thus,  $A_C(t, q) \leq S_C(t, q) \leq E_C(t, q)$ .  $\forall t \in [0, T]$ ,  $q \in \mathbb{R}_+$ .

## 5.4 Relationship between Price of Callable Sukūk and Ijārah Rate

### 5.4.1 Definitions

**Definition 5.4.1.** *Duration* is an approximate measure of the sensitivity of bond price to changes in interest rate. Mathematically, duration is the slope of the price/yield curve divided by the bond price. It is the first derivative of the bond price with respect to interest rates [20].

**Definition 5.4.2.** *Macaulay duration* is the weighted average maturity of cash flows and it is named under Frederick Macaulay who introduced this idea [13].

**Definition 5.4.3.** *Modified duration* is an extension of Macaulay duration which measure the sensitivity of bond price to changes in interest rate [20].

**Definition 5.4.4.** *Convexity* is an approximate measure of the non-linear relationship between price and yield. It is the second derivative of the bond price with respect to interest rates. Mathematically, it is the rate of change in the slope of price-yield curve divided by bond price [20].

Note that the concepts of duration, modified duration and convexity bond are used also for Sukūk. However, Sukūk use the Ijārah (rental) rate concept instead of interest rate for duration, modified duration and convexity. Also, during simulation graphs I will use the notions D and MD for duration and modified duration, respectively.

### 5.4.2 Callable Sukūk Duration and Convexity Derivations

Duration and convexity are two metrics used to help investors understand how the price of a callable Sukūk will be affected by changes in Ijārah (rental) rates. How a

callable Sukūk's price responds to changes in Ijārah rates is measured by its duration. Also, it can help the investors to understand the implications for a callable Sukūk's price should Ijārah rates change. The change in a callable Sukūk's duration for a given change in Ijārah can be measured by its convexity. In this section, we want to study the relationship between the price of callable Sukūk and Ijārah. These relationships are represented by the duration and convexity. Therefore, we want to find the first derivative of callable Sukūk formula to get the duration and find the second derivative to get the convexity. Recall the callable Sukūk price which is

$$\text{Price of callable Sukūk} = \text{Price of straight Sukūk} - \text{Call option price.} \quad (5.6)$$

$$C_S(t) = FV - (\mathbb{1}_{\{t \leq \frac{T}{2}\}} [S_0 e^{-q\tau_s} \Phi(d_{3q}(\tau_s)) - K e^{-r\tau_s} \Phi(d_{4q}(\tau_s))] + [S_0 e^{-q\tau} \Lambda_{1q}(\tau) - K e^{-r\tau} \Lambda_{2q}(\tau)]), \quad (5.7)$$

where, in this case,

$$d_{3q}(\tau_s) = \frac{\ln(S_0/K) + (r - q + \frac{1}{2}\sigma^2) \cdot \tau_s}{\sigma \sqrt{\tau_s}}, \quad (5.8)$$

$$d_{4q}(\tau_s) = \frac{\ln(S_0/K) + (r - q - \frac{1}{2}\sigma^2) \cdot \tau_s}{\sigma \sqrt{\tau_s}}, \quad (5.9)$$

$$d_{5q}(\beta, \tau) = \frac{\ln(S_0/K) + (r - q) \cdot \tau + \sigma\beta}{\sigma \sqrt{\frac{\tau}{2}}}, \quad (5.10)$$

$$d_{6q}(\beta, \tau) = \frac{\ln(S_0/K) + (r - q - \frac{1}{2}\sigma^2) \cdot \tau + \sigma\beta}{\sigma \sqrt{\frac{\tau}{2}}}, \quad (5.11)$$

$$\Lambda_{1q}(\tau) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi\tau}} e^{-\frac{\beta^2}{\tau}} e^{-\frac{1}{4}\sigma^2\tau + \sigma\beta} \Phi(d_{5q}(\beta, \tau)) d\beta, \quad (5.12)$$

$$\Lambda_{2q}(\tau) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi\tau}} e^{-\frac{\beta^2}{\tau}} \Phi(d_{6q}(\beta, \tau)) d\beta. \quad (5.13)$$

$$(5.14)$$

The duration is the first derivative of the formula (5.7) with respect to  $r$ . We

can see that we will need to compute the derivatives of  $\Phi(d_{3q})$ ,  $\Phi(d_{4q})$ ,  $\Lambda_{1q}$ , and  $\Lambda_{2q}$ , with respect to  $r$ . Since we are interested in the derivative with respect to  $r$ , we will consider all other variables  $\tau_s, \tau, \sigma, \beta$  as constants. All of the expressions on  $S_C$  formula depend on  $r, \tau_s, \tau, \sigma, \beta$ , etc. So, instead of writing  $S_C(r; \tau_s, \tau, \sigma, \beta)$  and  $d_{3q}(r; \tau_s, \tau, \sigma, \beta)$ , etc., we only use the notations  $S_C(r)$ ,  $d_{3q}(r)$  to emphasize that the independent variable is  $r$  and avoid writing very long expressions. Also, when  $d_{5q}, d_{6q}$ , etc appear inside the integral sign, we write  $d_{5q}(\beta)$  to emphasize that the variable of integration is  $\beta$ .

Recall that

$$\Phi(\chi) = \int_{-\chi}^{\infty} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz. \quad (5.15)$$

Thus, by the fundamental theorem of calculus, we have that

$$\frac{d\Phi}{d\chi} = \frac{d}{d\chi} \int_{-\chi}^{\infty} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz = \frac{e^{-\frac{\chi^2}{2}}}{\sqrt{2\pi}}. \quad (5.16)$$

We also know that

$$\Lambda_{1q} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi\tau}} e^{-\frac{\beta^2}{\tau} - \frac{\sigma^2}{4}\tau + \sigma\beta} \cdot \Phi(d_{5q}(\beta, \tau, r)) d\beta. \quad (5.17)$$

Hence, by Leibniz's rule for differentiation under the integral sign, we get

$$\begin{aligned} \frac{d\Lambda_{1q}}{dr} &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi\tau}} e^{-\frac{\beta^2}{\tau} - \frac{\sigma^2}{4}\tau + \sigma\beta} \cdot \frac{d}{dr} \Phi(d_{5q}(\beta, \tau, r)) d\beta \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi\tau}} e^{-\frac{\beta^2}{\tau} - \frac{\sigma^2}{4}\tau + \sigma\beta} \cdot \Phi'(d_{5q}(\beta, \tau, r)) \frac{d}{dr} d_{5q}(\beta, \tau, r) d\beta \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi\tau}} e^{-\frac{\beta^2}{\tau} - \frac{\sigma^2}{4}\tau + \sigma\beta} \cdot \frac{e^{-\frac{1}{2}(d_{5q}(\beta, \tau, r))^2}}{\sqrt{2\pi}} \cdot \frac{\sqrt{2\tau}}{\sigma} d\beta \\ &= \int_{-\infty}^{\infty} \frac{1}{\pi\sigma} e^{-\frac{\beta^2}{\tau} - \frac{\sigma^2}{4}\tau + \sigma\beta - \frac{(d_{5q}(\beta, \tau, r))^2}{2}} d\beta. \end{aligned} \quad (5.18)$$

Since

$$\Lambda_{2q} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi\tau}} e^{-\frac{\beta^2}{\tau}} \Phi(d_{6q}(\beta, \tau, r)) d\beta, \quad (5.19)$$

then, by Leibniz's rule, we get

$$\begin{aligned}
\frac{d\Lambda_{2q}}{dr} &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi\tau}} e^{-\frac{\beta^2}{\tau}} \frac{d}{dr} \Phi(d_{6q}(\beta, \tau, r)) d\beta \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi\tau}} e^{-\frac{\beta^2}{\tau}} \Phi'(d_{6q}(\beta, \tau, r)) \frac{d}{dr} d_{6q}(\beta, \tau, r) d\beta \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi\tau}} e^{-\frac{\beta^2}{\tau}} \frac{e^{-\frac{(d_{6q}(\beta, \tau, r))^2}{2}}}{\sqrt{2\pi}} \frac{\sqrt{2\tau}}{\sigma} d\beta \\
&= \int_{-\infty}^{\infty} \frac{1}{\pi\sigma} e^{-\frac{\beta^2}{\tau} - \frac{(d_{6q}(\beta, \tau, r))^2}{2}} d\beta.
\end{aligned} \tag{5.20}$$

We assume that  $t \leq \frac{T}{2}$  in the following derivations for obtaining the expressions for duration and convexity. For the case  $t > \frac{T}{2}$ , we can substitute  $\tau_s = 0$  in the following final expressions to get the corresponding expressions for duration and convexity.

Differentiating (5.7) with respect to  $r$  and assuming all other parameters are constants, we get

$$\begin{aligned}
\frac{d(S_C)}{dr} &= -S_0 e^{-q\tau_s} \frac{d}{dr} \Phi(d_{3q}(r)) + K(-\tau_s) e^{-r\tau_s} \Phi(d_{4q}(r)) + K e^{-r\tau_s} \frac{d}{dr} \Phi(d_{4q}(r)) \\
&\quad - S_0 e^{-q\tau} \frac{d}{dr} \Lambda_{1q}(r) + \tau(-K) e^{-r\tau} \Lambda_{2q}(r) + K e^{-r\tau} \frac{d}{dr} \Lambda_{2q}(r).
\end{aligned} \tag{5.21}$$

Using the derivative expressions in (5.16), we have (5.18) and (5.20),

$$\begin{aligned}
\frac{d(S_C)}{dr} &= -S_0 e^{-q\tau_s} \frac{e^{-\frac{(d_{3q}(r))^2}{2}}}{\sqrt{2\pi}} \frac{\sqrt{\tau_s}}{\sigma} - K\tau_s e^{-r\tau_s} \Phi(d_{4q}(r)) + K e^{-r\tau_s} \frac{e^{-\frac{(d_{4q}(r))^2}{2}}}{\sqrt{2\pi}} \frac{\sqrt{\tau_s}}{\sigma} \\
&\quad - S_0 e^{-q\tau} \int_{-\infty}^{\infty} \frac{1}{\pi\sigma} e^{-\frac{\beta^2}{\tau} - \frac{\sigma^2\tau}{4} + \sigma\beta - \frac{(d_{5q}(r))^2}{2}} d\beta - \tau K e^{-r\tau} \Lambda_{2q}(r) \\
&\quad + K e^{-r\tau} \int_{-\infty}^{\infty} \frac{1}{\pi\sigma} e^{-\frac{\beta^2}{\tau} - \frac{(d_{6q}(r))^2}{2}} d\beta.
\end{aligned} \tag{5.22}$$

Collecting similar terms, gives

$$\begin{aligned}
\frac{d(S_C)}{dr} = & -S_0 e^{-q\tau_s} \frac{\sqrt{\tau_s}}{\sigma\sqrt{2\pi}} e^{-\frac{(d_{3q})^2}{2}} - K\tau_s e^{-r\tau_s} \Phi(d_{4q}(r)) + \frac{K\sqrt{\tau_s}}{\sigma\sqrt{2\pi}} e^{-r\tau_s - \frac{(d_{4q})^2}{2}} \\
& - S_0 e^{-q\tau} \int_{-\infty}^{\infty} \frac{1}{\pi\sigma} e^{-\frac{\beta^2}{\tau} - \frac{\sigma^2\tau}{4} + \sigma\beta} e^{-\frac{(d_{5q}(r))^2}{2}} d\beta - \tau K e^{-r\tau} \Lambda_{2q}(r) \\
& + K e^{-r\tau} \int_{-\infty}^{\infty} \frac{1}{\pi\sigma} e^{-\frac{\beta^2}{\tau}} e^{-\frac{(d_{6q}(r))^2}{2}} d\beta.
\end{aligned} \tag{5.23}$$

If we want to find the modified duration, we divide the duration formula (5.23) by the expression  $1 + r$ . So, we can define the modified duration as

$$\text{Modified duration} = \frac{1}{(1 + r)} \cdot \frac{d(S_C)}{dr}. \tag{5.24}$$

If the callable Sukūk price changes almost linearly with respect to  $r$  then the duration can be used to approximate the callable Sukūk price around a fixed Ijārah rate  $r_0$  by using a linear Taylor polynomial whose slope is the duration. That is

$$S_C \approx S_C(r_0) + \frac{d(S_C)}{dr}(r_0)(r - r_0). \tag{5.25}$$

Similarly, modified duration can be used to approximate the callable Sukūk price around  $r_0$  by using a linear polynomial whose slope is the modified duration. That is

$$S_C \approx S_C(r_0) + \frac{1}{(1 + r_0)} \cdot \frac{d(S_C)}{dr}(r_0)(r - r_0). \tag{5.26}$$

If the price does not change linearly with respect to  $r$  then the expressions (5.25) and (5.26) are not good approximations of the callable Sukūk price. In other words, just duration and modified duration are not enough to give a complete picture of the the callable Sukūk price around a fixed Ijārah rate  $r_0$ . This is also clear from Figures 5.2 and 5.3.

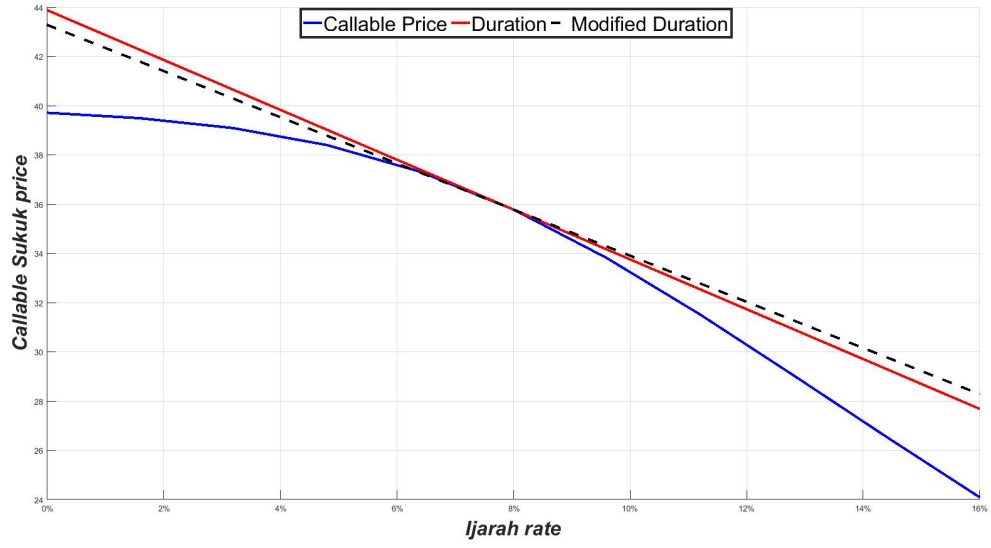


Figure 5.2: Duration and modified duration of callable Sukūk price-Ijārah relationship when  $t \leq \frac{T}{2}$

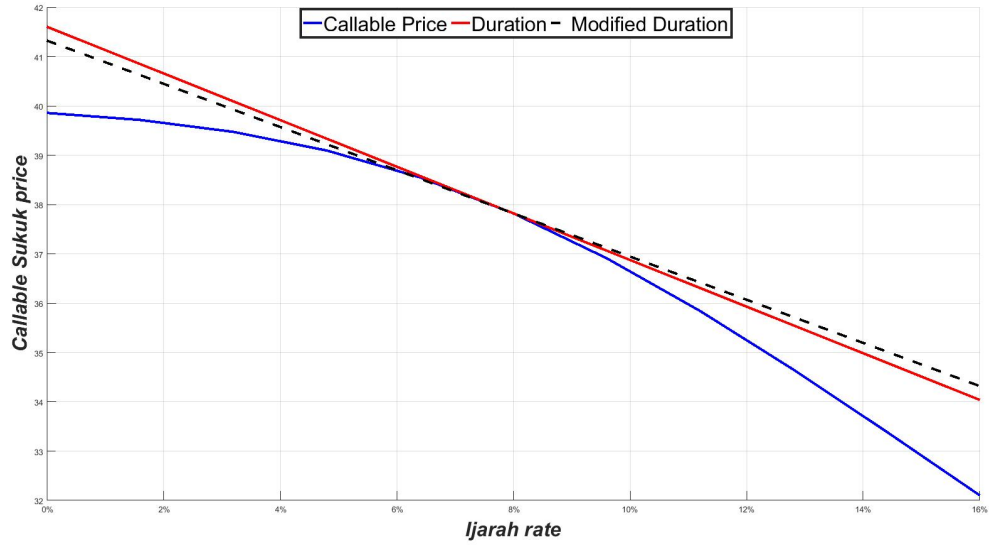


Figure 5.3: Duration and modified duration of callable Sukūk price-Ijārah relationship when  $t > \frac{T}{2}$

From Figures 5.2 and 5.3, we notice that the curvature of the callable Sukūk price-Ijārah rate curve remains same whether we consider the price-Ijārah rate curve before mid-term or after mid-term. As Figures 5.2 and 5.3 show, as we move away from



the tangential point the duration and modified duration move away from the callable Sukūk price. It is also clear from these figures that the callable Sukūk price-Ijārah rate curve is concave or negatively convex. Since the convexity can be estimated using the second derivative, we will need the second derivative of the callable Sukūk price with respect to Ijārah rate to get a better approximation and see the rate of change in the slope of price-Ijārah curve. So, differentiating  $\frac{d(SC)}{dr}$  with respect to  $r$ , we get

$$\begin{aligned}
\frac{d^2(SC)}{dr^2} = & -S_0 e^{-q\tau_s} \frac{\sqrt{\tau_s}}{\sigma} \frac{e^{-\frac{d_{3q}^2}{2}}}{\sqrt{2\pi}} (-d_{3q}) \frac{\sqrt{\tau_s}}{\sigma} + K \tau_s^2 e^{-r\tau_s} \Phi(d_{4q}(r)) \\
& - K \tau_s e^{-r\tau_s} \frac{e^{-\frac{d_{4q}^2}{2}}}{\sqrt{2\pi}} \frac{\sqrt{\tau_s}}{\sigma} + K (-\tau_s) e^{-r\tau_s} \frac{e^{-\frac{d_{4q}^2}{2}}}{\sqrt{2\pi}} \frac{\sqrt{\tau_s}}{\sigma} \\
& + K e^{-r\tau_s} \frac{e^{-\frac{d_{4q}^2}{2}}}{\sqrt{2\pi}} \frac{\sqrt{\tau_s}}{\sigma} (-d_{4q}) \frac{\sqrt{\tau_s}}{\sigma} \\
& - S_0 e^{-q\tau} \int_{-\infty}^{\infty} \frac{1}{\pi\sigma} e^{-\frac{\beta^2}{\tau} - \frac{\sigma^2\tau}{4} + \sigma\beta} e^{-\frac{d_{5q}^2}{2}} (-d_{5q}) \frac{\sqrt{2\tau}}{\sigma} d\beta \\
& - \tau K (-\tau) e^{-r\tau} \Lambda_2 - \tau K e^{-r\tau} \int_{-\infty}^{\infty} \frac{1}{\pi\sigma} e^{-\frac{\beta^2}{\tau}} e^{-\frac{d_{6q}^2}{2}} (-d_{6q}) \frac{\sqrt{2\tau}}{\sigma} d\beta \\
& - \tau K e^{-r\tau} \int_{-\infty}^{\infty} \frac{1}{\pi\sigma} e^{-\frac{\beta^2}{\tau}} e^{-\frac{d_{6q}^2}{2}} d\beta + K e^{-r\tau} \int_{-\infty}^{\infty} \frac{1}{\pi\sigma} e^{-\frac{\beta^2}{\tau}} e^{-\frac{d_{6q}^2}{2}} (-d_{6q}) \frac{\sqrt{2\tau}}{\sigma} d\beta.
\end{aligned} \tag{5.27}$$

After simplifying, we get

$$\begin{aligned}
\frac{d^2(SC)}{dr^2} = & S_0 e^{-q\tau_s} \frac{\tau_s}{\sigma^2} \frac{e^{-\frac{d_{3q}^2}{2}}}{\sqrt{2\pi}} d_{3q} + K \tau_s^2 e^{-r\tau_s} \Phi(d_{4q}) - \frac{K \tau_s \sqrt{\tau_s}}{\sigma} e^{-r\tau_s} \frac{e^{-\frac{d_{4q}^2}{2}}}{\sqrt{2\pi}} \\
& - \frac{K \tau_s \sqrt{\tau_s}}{\sigma} e^{-r\tau_s} \frac{e^{-\frac{d_{4q}^2}{2}}}{\sqrt{2\pi}} - \frac{K \tau_s}{\sigma^2} e^{-r\tau_s} \frac{e^{-\frac{d_{4q}^2}{2}}}{\sqrt{2\pi}} d_{4q} \\
& + S_0 e^{-q\tau} \frac{\sqrt{2\tau}}{\pi\sigma^2} \int_{-\infty}^{\infty} e^{-\frac{\beta^2}{\tau} - \frac{\sigma^2\tau}{4} + \sigma\beta} e^{-\frac{d_{5q}^2}{2}} d_{5q} d\beta + \tau^2 K e^{-r\tau} \Lambda_{2q} \\
& + \tau K e^{-r\tau} \frac{\sqrt{2\tau}}{\pi\sigma^2} \int_{-\infty}^{\infty} e^{-\frac{\beta^2}{\tau}} e^{-\frac{d_{6q}^2}{2}} d_{6q} d\beta \\
& - \frac{\tau K e^{-r\tau}}{\pi\sigma} \int_{-\infty}^{\infty} e^{-\frac{\beta^2}{\tau}} e^{-\frac{d_{6q}^2}{2}} d\beta - \frac{K \sqrt{2\tau}}{\pi\sigma^2} e^{-r\tau} \int_{-\infty}^{\infty} e^{-\frac{\beta^2}{\tau}} e^{-\frac{d_{6q}^2}{2}} d_{6q} d\beta.
\end{aligned} \tag{5.28}$$

After collecting similar terms, we get

$$\begin{aligned}
\frac{d^2(S_C)}{dr^2} = & S_0 e^{-q\tau_s} \frac{\tau_s}{\sigma^2 \sqrt{2\pi}} d_{3q} e^{-\frac{1}{2}d_{3q}^2} \\
& + \left( K\tau_s^2 \Phi(d_{4q}) - \frac{2K\sigma\tau_s\sqrt{\tau_s} + K\tau_s d_{4q}}{\sigma^2 \sqrt{2\pi}} e^{-\frac{1}{2}d_{4q}^2} \right) e^{-r\tau_s} \\
& + S_0 e^{-q\tau} \frac{\sqrt{2\tau}}{\pi\sigma^2} \int_{-\infty}^{\infty} e^{-\frac{\beta^2}{\tau} - \frac{\sigma^2\tau}{4} + \sigma\beta} e^{-\frac{1}{2}(d_{5q})^2} d_{5q}(\beta) d\beta \\
& + K e^{-r\tau} \left( \tau^2 \Lambda_{2q} + \frac{(\tau-1)\sqrt{2\tau}}{\pi\sigma^2} \int_{-\infty}^{\infty} e^{-\frac{\beta^2}{\tau}} e^{-\frac{1}{2}d_{6q}^2} d_{6q}(\beta) d\beta - \frac{\tau}{\pi\sigma} \int_{-\infty}^{\infty} e^{-\frac{\beta^2}{\tau}} e^{-\frac{1}{2}d_{6q}^2} d\beta \right).
\end{aligned} \tag{5.29}$$

We can now approximate the callable Sukūk price by using its duration and convexity to build a second order Taylor polynomial as follows

$$S_C \approx S_C(r_0) + \frac{d(S_C)}{dr}(r_0)(r - r_0) + \frac{1}{2} \frac{d^2(S_C)}{dr^2}(r_0)(r - r_0)^2. \tag{5.30}$$

Also, the approximation of the callable Sukūk price by using its modified duration and convexity to build a second order polynomial as follows

$$S_C \approx S_C(r_0) + \frac{1}{1+r_0} \left( \frac{d(S_C)}{dr}(r_0)(r - r_0) + \frac{1}{2} \frac{d^2(S_C)}{dr^2}(r_0)(r - r_0)^2 \right). \tag{5.31}$$

The quadratic terms in Equations (5.30), (5.31) are called the convexity estimate by duration and modified duration respectively. As we can see in Figures 5.4 and 5.5 that adding the convexity estimate gives a better approximation to the callable Sukūk price, especially when the curvature of the price is not very pronounced. When the callable Sukūk price has a large curvature, the convexity estimate provides a good approximation only for small intervals around a specific Iājārah rate  $r_0$ . The price and yield of Ijārah are inversely related but not in a linear fashion. This is also clear from Figure 5.4 and 5.5 that if the Ijārah rate of Sukuk fall, then its price will increase and vice versa. These plots give us the sensitivity of callable Sukūk price to change in Ijārah rate. It must be

mentioned that before mid-term, the callable Sukūk price is more sensitive to change in Ijārah than after mid-term. This can be justified from Figures 5.4 and 5.5 where one can observe that the Sukūk price-Ijārah curve is more negatively convex in Figure 5.4 (that is, at a time before mid-term) than in Figure 5.5 (that is, at a time after mid-term).

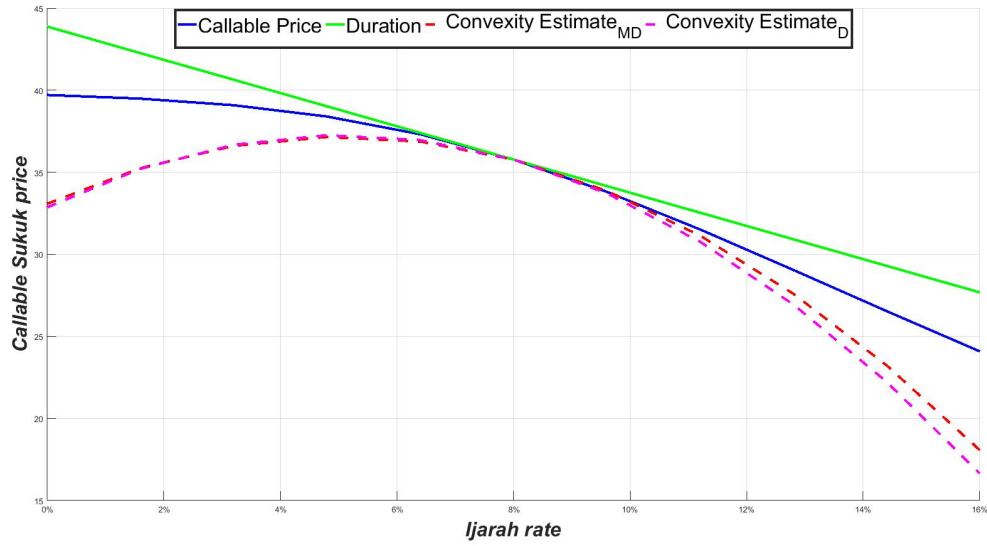


Figure 5.4: Convexity of callable Sukūk price-Ijārah relationship estimated by duration and modified duration when  $t \leq \frac{T}{2}$

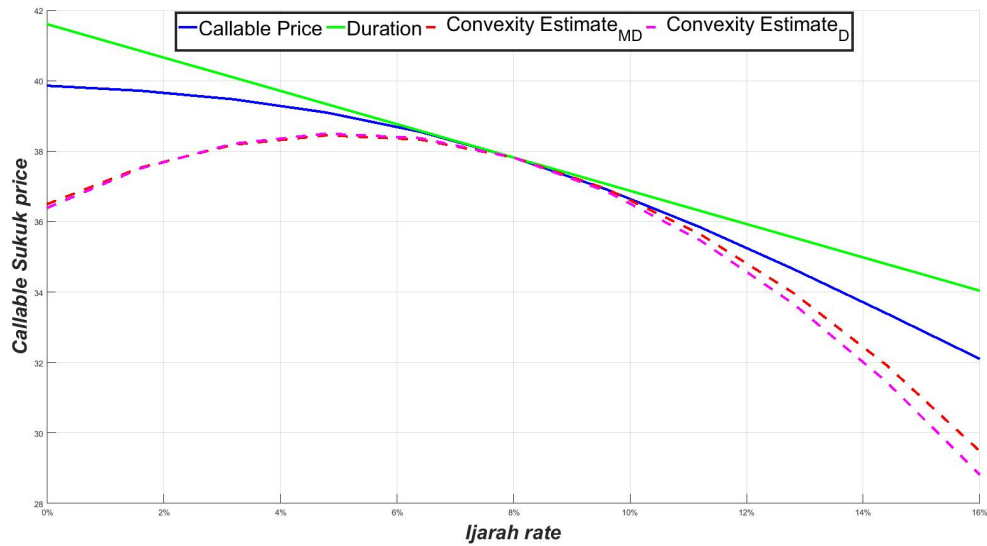


Figure 5.5: Convexity of callable Sukūk price-Ijārah relationship estimated by duration and modified duration when  $t > \frac{T}{2}$

The negative convexity of the callable Sukūk price-Ijārah curve is due to the fact that as the Ijārah rate decreases, the chance that the issuer of Sukūk will call the Sukūk increases. Therefore the Sukūk price does not increase with the same rate as that for the non-callable Sukūk. This results in concavity in the callable Sukūk price-Ijārah curve and hence in negative convexity.

Both the callable Sukūk and the callable bond have negative convexity. It was mentioned that at a time before mid-term, the callable Sukūk price-Ijārah curve is more curved than at a time after mid-term. If we compare with the callable bonds, this is in line with the fact that the callable bonds with longer maturity are more negatively convex than the callable bonds with less maturity [23]. For a callable Sukūk as the time to maturity decreases, the Sukūk price-Ijārah curve becomes more flat.

## 5.5 A Mid-term Puttable Sukūk

### 5.5.1 Definition

A puttable Sukūk is Sukūk that allows the holder of puttable to demand to repay the principal early at mid-term of maturity. Therefore, a puttable Sukūk is a straight bond embedded with mid-term put option of Ijārah Sukūk [51].

### 5.5.2 Statement of the Problem

As known in the American puttable the holder of puttable bond can demand to repay the principal early at any time before maturity [54]. On the other hand, the puttable Sukūk holder can demand to repay the principal early at mid-term of maturity. Therefore, we aim to find the puttable Sukūk price based on the mid-term put option of Sukūk that we derived in previous chapter. Similarly, we will find the price of puttable Sukūk. Then, we will compare the puttable Sukūk with the price of the corresponding American puttable bonds during lifespan. Moreover, we study the relationship between puttable price and the Iājarah rate by derive the duration and convexity formulas.

### 5.5.3 Assumptions and Notations

We use the same assumptions and notations as in the previous chapters and adding to these, the notations for this chapter  $S_P$  and  $A_P$  which are the puttable Sukūk and American puttable bond respectively.

### 5.5.4 Puttable Sukūk Price

A puttable Sukūk is a straight Sukūk embedded with mid-term put option of Sukūk [51]. The price of straight Sukūk is defined at par value. If the puttable Sukūk does not exceed the exercise price at mid-term, the holder will not exercise the option. Otherwise, the holder will use its right to sell the Sukūk because the non-puttable Sukūk price is equal to the puttable Sukūk price. Hence, the puttable Sukūk price will be higher than the straight Sukūk price because the put option give more value to the investor and the yield on puttable Sukūk is lower than the yield on straight Sukūk. Therefore, the price of puttable Sukūk with continuous Ijārah at time  $t$  is

$$\text{Price of puttable Sukūk} = \text{Price of straight Sukūk} + \text{Put option price}, \quad (5.32)$$

$$S_P(t) = FV + (\mathbb{1}_{\{t \leq \frac{T}{2}\}} [K e^{-r\tau_s} \Phi(-d_{4q}(\tau_s)) - S_0 e^{-q\tau_s} \Phi(-d_{3q}(\tau_s))] + [K e^{-r\tau} \Lambda_{2q}(\tau) - S_0 e^{-q\tau} \Lambda_{1q}(\tau)]). \quad (5.33)$$

Similar to callable Sukūk section, we assume that the straight Sukūk price is the face value or par value and find the put option price as in Chapter 4.

### 5.5.5 American Puttable Bond Price

The American puttable bond is a bond that allows the holder of puttable to demand the bond at any time before maturity. Therefore, the price of American puttable bond is always higher than the straight bond price [28]. The formula of puttable bond price is

$$\text{Price of American puttable bond} = \text{Price of straight bond} + \text{Put option price}. \quad (5.34)$$

In this section, we assume that the straight bond price is the face value and find the American put option price as in Chapter 4 by the approximation methods.

## 5.6 Comparison of Puttable Sukūk with American Puttable Bonds

### 5.6.1 Puttable Sukūk with Ijārah Compared to American Puttable Bond Price with Coupon at Time $t$

**Lemma 5.6.1.** Suppose the underlying asset pays Ijārah  $q$  for puttable Sukūk and coupon  $q$  for American puttable bond with the same strike price  $K$  and expiry time  $T$ . The price of the American puttable bond is greater than the price of puttable Sukūk,  $S_P(t, q) \leq A_P(t, q) \forall t \in [0, T]$ ,  $q \in \mathbb{R}_+$ .

*Proof.* Assume that the straight price or face value is the same and Ijārah  $q$  is equal coupon  $q$ . Also, as proved in Lemma 4.6.3 that the price of Sukūk put option is no greater than the price of the American put option for all time  $t$ . Therefore, the price of straight bond plus the put option price for Sukūk is no grater than in American puttable. Thus  $S_P(t, q) \leq A_P(t, q) \forall t \in [0, T]$ ,  $q \in \mathbb{R}_+$ .  $\square$

## 5.7 Simulations and discussions

In this section, we want to simulate the puttable Sukūk price versus the American puttable bond and see the prices' behavior over lifespan. We assume that the straight Sukūk price is the face value or par value and find the put option price as in previous chapters.

### 5.7.1 Puttable Sukūk Price with Ijārah Compared to American Puttable Bond Price with Coupon at Time t

In this step, we plotted the puttable Sukūk price over lifespan with Ijārah and compared it to American puttable bond price with coupon.

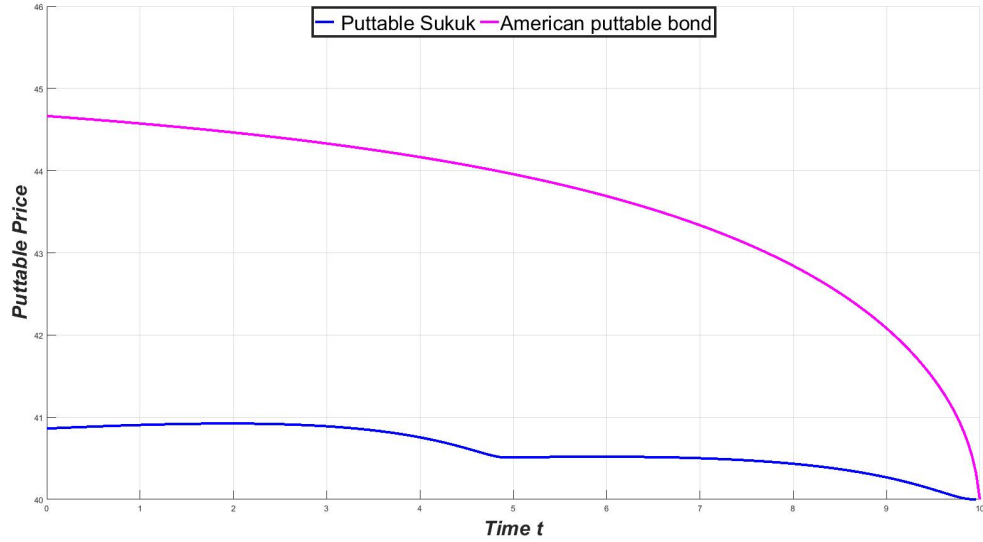


Figure 5.6: American puttable bond price with coupon compared to puttable Sukūk with Ijārah at time t.

Figure 5.6 shows that the price of Sukūk puttable is not greater than the price of the American puttable bond as we have proved in Lemma 5.6.1 . Thus,  $S_P(t, q) \leq A_P(t, q)$ .  $\forall t \in [0, T]$  ,  $q \in \mathbb{R}_+$ .



## 5.8 Relationship between Price of Puttable Sukūk and Ijārah Rate

Similar to Section 5.4, we want to study the relationship between the price of puttable Sukūk and Ijārah. Therefore, we want to find the duration by the first derivative of the puttable Sukūk price formula and the convexity by the second derivative. Hence, recall the puttable Sukūk price which is

$$S_P(t) = FV + (\mathbb{1}_{\{t \leq \frac{T}{2}\}} [K e^{-r\tau_s} \Phi(-d_{4q}(\tau_s)) - S_0 e^{-q\tau_s} \Phi(-d_{3q}(\tau_s))] + [K e^{-r\tau} \Lambda_{2q}(\tau) - S_0 e^{-q\tau} \Lambda_{1q}(\tau)]), \quad (5.35)$$

where

$$d_{3q}(\tau_s) = \frac{\ln(S_0/K) + (r - q + \frac{1}{2}\sigma^2) \cdot \tau_s}{\sigma \sqrt{\tau_s}}, \quad (5.36)$$

$$d_{4q}(\tau_s) = \frac{\ln(S_0/K) + (r - q - \frac{1}{2}\sigma^2) \cdot \tau_s}{\sigma \sqrt{\tau_s}}, \quad (5.37)$$

$$d_{5q}(\beta, \tau) = \frac{\ln(S_0/K) + (r - q)\tau + \sigma\beta}{\sigma \sqrt{\frac{\tau}{2}}}, \quad (5.38)$$

$$d_{6q}(\beta, \tau) = \frac{\ln(S_0/K) + (r - q - \frac{1}{2}\sigma^2)\tau + \sigma\beta}{\sigma \sqrt{\frac{\tau}{2}}}, \quad (5.39)$$

$$\Lambda_{1q}(\tau) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi\tau}} e^{\frac{-\beta^2}{\tau}} e^{-\frac{1}{4}\sigma^2\tau + \sigma\beta} \Phi(-d_{5q}(\beta, \tau)) d\beta, \quad (5.40)$$

$$\Lambda_{2q}(\tau) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi\tau}} e^{-\frac{\beta^2}{\tau}} \Phi(-d_{6q}(\beta, \tau)) d\beta. \quad (5.41)$$

The duration of the puttable Sukūk is the first derivative of the puttable Sukūk price with respect to Ijārah rate  $r$ . From the (5.35) we can see that we will need to compute the derivatives of  $\Phi(d_{3q})$ ,  $\Phi(d_{4q})$ ,  $\Lambda_{1q}$ , and  $\Lambda_{2q}$ , with respect to  $r$ . Since we are interested in the derivative with respect to  $r$  we will consider all other variables  $\tau_s, \tau, \sigma, \beta$

as constants. All of the expressions on  $S_P$  formula depend on  $r, \tau_s, \tau, \sigma, \beta$ , etc. So, instead of writing  $S_P(r; \tau_s, \tau, \sigma, \beta)$  and  $d_{3q}(r; \tau_s, \tau, \sigma, \beta)$ , etc, we only use the notations  $S_P(r), d_{3q}(r)$  to emphasize that the independent variable is  $r$  and avoid writing very long expressions. Also, when  $d_{5q}, d_{6q}$ , etc appear inside the integral sign, we write  $d_{5q}(r, \beta)$  to emphasize that the variable of integration is  $\beta$ .

Recall that

$$\Phi(\chi) = \int_{-\infty}^{\chi} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz. \quad (5.42)$$

Thus, by the fundamental theorem of calculus, we have that

$$\frac{d\Phi}{d\chi} = \frac{d}{d\chi} \int_{-\infty}^{\chi} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz = \frac{e^{-\frac{\chi^2}{2}}}{\sqrt{2\pi}}. \quad (5.43)$$

We also know that

$$\Lambda_{1q} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi\tau}} e^{-\frac{\beta^2}{\tau}} e^{-\frac{1}{4}\sigma^2\tau + \sigma\beta} \Phi(-d_{5q}(\beta, \tau, r)) d\beta. \quad (5.44)$$

Hence, by Leibniz's rule for differentiation under the integral sign, we get

$$\begin{aligned} \frac{d\Lambda_{1q}}{dr} &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi\tau}} e^{-\frac{\beta^2}{\tau}} e^{-\frac{1}{4}\sigma^2\tau + \sigma\beta} \cdot \frac{d}{dr} \Phi(-d_{5q}(\beta, \tau, r)) d\beta \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi\tau}} e^{-\frac{\beta^2}{\tau}} e^{-\frac{1}{4}\sigma^2\tau + \sigma\beta} \cdot \Phi'(-d_{5q}(\beta, \tau, r)) \frac{d}{dr} (-d_{5q}(\beta, \tau, r)) d\beta \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi\tau}} e^{-\frac{\beta^2}{\tau}} e^{-\frac{1}{4}\sigma^2\tau + \sigma\beta} \cdot \frac{e^{-\frac{1}{2}(-d_{5q}(\beta, \tau, r))^2}}{\sqrt{2\pi}} \cdot \frac{-\sqrt{2\tau}}{\sigma} d\beta \\ &= - \int_{-\infty}^{\infty} \frac{1}{\pi\sigma} e^{-\frac{\beta^2}{\tau} - \frac{\sigma^2}{4}\tau + \sigma\beta - \frac{(d_{5q}(\beta, \tau, r))^2}{2}} d\beta. \end{aligned} \quad (5.45)$$

Since

$$\Lambda_{2q} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi\tau}} e^{-\frac{\beta^2}{\tau}} \Phi(-d_{6q}(\beta, \tau, r)) d\beta, \quad (5.46)$$

then, by Leibniz's rule, we get

$$\begin{aligned}
\frac{d\Lambda_{2q}}{dr} &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi\tau}} e^{-\frac{\beta^2}{\tau}} \frac{d}{dr} \Phi(-d_{6q}(\beta, \tau, r)) d\beta \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi\tau}} e^{-\frac{\beta^2}{\tau}} \Phi'(-d_{6q}(\beta, \tau, r)) \frac{d}{dr} (-d_{6q}(\beta, \tau, r)) d\beta \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi\tau}} e^{-\frac{\beta^2}{\tau}} \frac{e^{-\frac{(-d_{6q}(\beta, \tau, r))^2}{2}} - \sqrt{2\tau}}{\sqrt{2\pi}} \frac{1}{\sigma} d\beta \\
&= - \int_{-\infty}^{\infty} \frac{1}{\pi\sigma} e^{-\frac{\beta^2}{\tau} - \frac{(d_{6q}(\beta, \tau, r))^2}{2}} d\beta.
\end{aligned} \tag{5.47}$$

Thus, differentiating (5.35) with respect to  $r$  and assuming all other parameters are constants, we get

$$\begin{aligned}
\frac{d(S_P)}{dr} &= K(-\tau_s) e^{-r\tau_s} \Phi(-d_{4q}(r)) + K e^{-r\tau_s} \frac{d}{dr} \Phi(-d_{4q}(r)) - S_0 e^{-q\tau_s} \frac{d}{dr} \Phi(-d_{3q}(r)) \\
&\quad + K(-\tau) e^{-r\tau} \Lambda_{2q}(r) + K e^{-r\tau} \frac{d}{dr} \Lambda_{2q}(r) - S_0 e^{-q\tau} \frac{d}{dr} \Lambda_{1q}(r).
\end{aligned} \tag{5.48}$$

Using the derivative expressions in (5.43), (5.45) and (5.47), we have

$$\begin{aligned}
\frac{d(S_P)}{dr} &= -K\tau_s e^{-r\tau_s} \Phi(-d_{4q}(r)) + K e^{-r\tau_s} \frac{e^{-\frac{(-d_{4q}(r))^2}{2}} - \sqrt{\tau_s}}{\sqrt{2\pi}} \frac{1}{\sigma} - S_0 e^{-q\tau_s} \frac{e^{-\frac{(-d_{3q}(r))^2}{2}} - \sqrt{\tau_s}}{\sqrt{2\pi}} \frac{1}{\sigma} \\
&\quad - K\tau e^{-r\tau} \Lambda_{2q}(r) + K e^{-r\tau} \left( - \int_{-\infty}^{\infty} \frac{1}{\pi\sigma} e^{-\frac{\beta^2}{\tau} - \frac{(-d_{6q}(r))^2}{2}} d\beta \right) \\
&\quad - S_0 e^{-q\tau} \left( - \int_{-\infty}^{\infty} \frac{1}{\pi\sigma} e^{-\frac{\beta^2}{\tau} - \frac{\sigma^2\tau}{4} + \sigma\beta - \frac{(-d_{5q}(r))^2}{2}} d\beta \right).
\end{aligned} \tag{5.49}$$

Collecting similar terms, gives

$$\begin{aligned}
\frac{d(S_P)}{dr} &= -K\tau_s e^{-r\tau_s} \Phi(-d_{4q}(r)) - \frac{K\sqrt{\tau_s}}{\sigma\sqrt{2\pi}} e^{-r\tau_s - \frac{(-d_{4q}(r))^2}{2}} + S_0 e^{-q\tau_s} \frac{\sqrt{\tau_s}}{\sigma\sqrt{2\pi}} e^{-\frac{(-d_{3q}(r))^2}{2}} \\
&\quad - K\tau e^{-r\tau} \Lambda_{2q}(r) - K e^{-r\tau} \int_{-\infty}^{\infty} \frac{1}{\pi\sigma} e^{-\frac{\beta^2}{\tau}} e^{-\frac{(-d_{6q}(r))^2}{2}} d\beta \\
&\quad + S_0 e^{-q\tau} \int_{-\infty}^{\infty} \frac{1}{\pi\sigma} e^{-\frac{\beta^2}{\tau} - \frac{\sigma^2\tau}{4} + \sigma\beta} e^{-\frac{(-d_{5q}(r))^2}{2}} d\beta.
\end{aligned} \tag{5.50}$$

Similarly as in Section 5.4 if we want to find the modified duration, we divide the duration formula (5.50) by the expression  $1 + r$ . So, we can define the modified duration as

$$\text{Modified duration} = \frac{1}{(1 + r)} \cdot \frac{d(S_P)}{dr}. \quad (5.51)$$

If the puttable Sukūk price changes almost linearly with respect to  $r$  then the duration can be used to approximate the puttable Sukūk price around a fixed Ijārah rate  $r_0$  by using a linear Taylor polynomial whose slope is the duration. That is

$$S_P \approx S_P(r_0) + \frac{d(S_P)}{dr}(r_0)(r - r_0). \quad (5.52)$$

Similarly, modified duration can be used to approximate the puttable Sukūk price by using a linear polynomial whose slope is the modified duration. That is

$$S_P \approx S_P(r_0) + \frac{1}{(1 + r_0)} \cdot \frac{d(S_P)}{dr}(r_0)(r - r_0). \quad (5.53)$$

If the price does not change linearly with respect to  $r$  then the expression (5.52) and (5.53) are not good approximations of the puttable Sukūk price. In other words, just duration and modified duration are not enough to give a complete picture of the the puttable Sukūk price around a fixed Ijārah rate  $r_0$ . This is also clear from Figures 5.7 and 5.8.

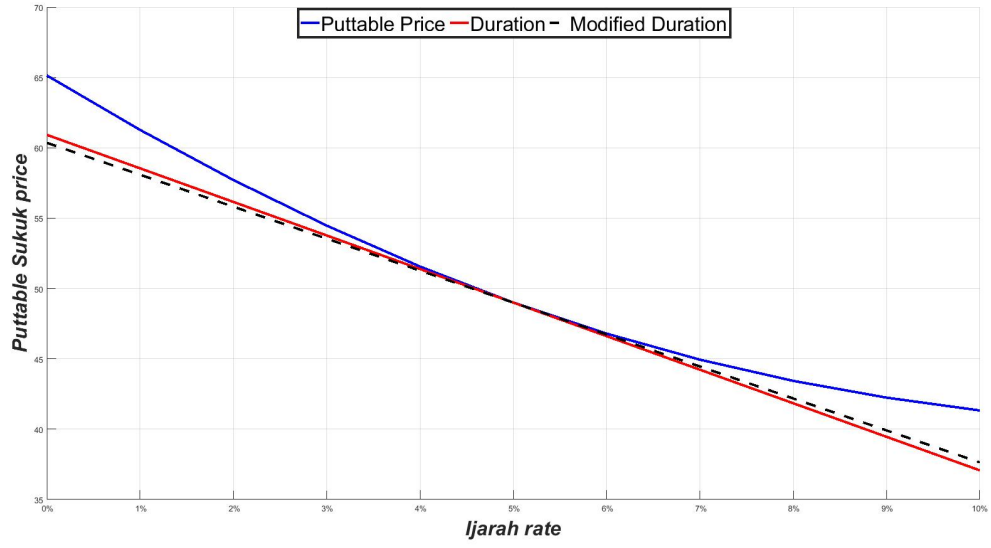


Figure 5.7: Duration and modified duration of puttable Sukūk price-Ijārah relationship when  $t \leq \frac{T}{2}$

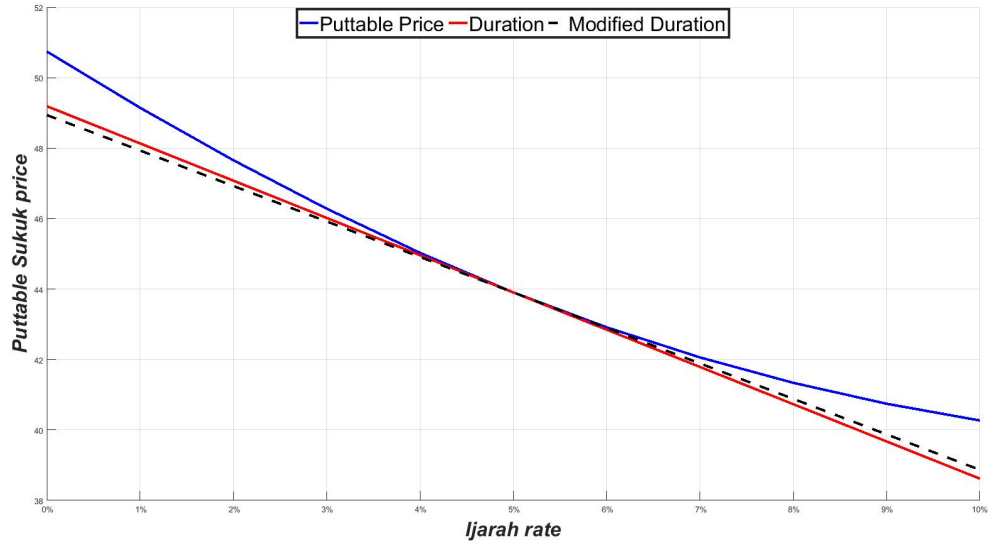


Figure 5.8: Duration and modified duration of puttable Sukūk price-Ijārah relationship when  $t > \frac{T}{2}$

From Figures 5.7 and 5.8, we notice that the curvature of the puttable Sukūk price-Ijārah rate curve is more positively convex before mid-term. As Figures 5.7 and 5.8 show, as we move away from the tangential point the duration and modified duration

move away from the puttable Sukūk price. It is also clear from these figures that the puttable Sukūk price-Ijārah rate curve is positively convex. Since the convexity can be estimated using the second derivative, we will need the second derivative of the puttable Sukūk price with respect to Ijārah rate to get a better approximation and see the rate of change in the slope of price-Ijārah curve. So, differentiating  $\frac{d(S_P)}{dr}$  with respect to  $r$ , we get

$$\begin{aligned}
\frac{d^2(S_P)}{dr^2} = & K\tau_s^2 e^{-r\tau_s} \Phi(-d_{4q}(r)) - K\tau_s e^{-r\tau_s} \frac{e^{-\frac{(-d_{4q}(r))^2}{2}}}{\sqrt{2\pi}} \frac{-\sqrt{\tau_s}}{\sigma} \\
& - K(-\tau_s) e^{-r\tau_s} \frac{e^{-\frac{d_{4q}^2}{2}}}{\sqrt{2\pi}} \frac{\sqrt{\tau_s}}{\sigma} - K e^{-r\tau_s} \frac{e^{-\frac{d_{4q}^2}{2}}}{\sqrt{2\pi}} \frac{\sqrt{\tau_s}}{\sigma} (-d_{4q}) \frac{\sqrt{\tau_s}}{\sigma} \\
& + S_0 e^{-q\tau_s} \frac{\sqrt{\tau_s}}{\sigma} \frac{e^{-\frac{d_{3q}^2}{2}}}{\sqrt{2\pi}} (-d_{3q}) \frac{\sqrt{\tau_s}}{\sigma} \\
& - K\tau(-\tau) e^{-r\tau} \Lambda_{2q} - K\tau e^{-r\tau} \left( - \int_{-\infty}^{\infty} \frac{1}{\pi\sigma} e^{-\frac{\beta^2}{\tau}} e^{-\frac{d_{6q}^2}{2}} d\beta \right) \\
& - K(-\tau) e^{-r\tau} \int_{-\infty}^{\infty} \frac{1}{\pi\sigma} e^{-\frac{\beta^2}{\tau}} e^{-\frac{d_{6q}^2}{2}} d\beta - K e^{-r\tau} \int_{-\infty}^{\infty} \frac{1}{\pi\sigma} e^{-\frac{\beta^2}{\tau}} e^{-\frac{d_{6q}^2}{2}} (-d_{6q}) \frac{\sqrt{2\tau}}{\sigma} d\beta \\
& + S_0 e^{-q\tau} \int_{-\infty}^{\infty} \frac{1}{\pi\sigma} e^{-\frac{\beta^2}{\tau} - \frac{\sigma^2\tau}{4} + \sigma\beta} e^{-\frac{d_{5q}^2}{2}} (-d_{5q}) \frac{\sqrt{2\tau}}{\sigma} d\beta. \tag{5.54}
\end{aligned}$$

After simplifying, we get

$$\begin{aligned}
\frac{d^2(S_P)}{dr^2} = & K\tau_s^2 e^{-r\tau_s} \Phi(-d_{4q}(r)) + \frac{K\tau_s \sqrt{\tau_s}}{\sigma} e^{-r\tau_s} \frac{e^{-\frac{(-d_{4q}(r))^2}{2}}}{\sqrt{2\pi}} \\
& + \frac{K\tau_s \sqrt{\tau_s}}{\sigma} e^{-r\tau_s} \frac{e^{-\frac{d_{4q}^2}{2}}}{\sqrt{2\pi}} + \frac{K\tau_s}{\sigma^2} e^{-r\tau_s} \frac{e^{-\frac{d_{4q}^2}{2}}}{\sqrt{2\pi}} d_{4q} - S_0 e^{-q\tau_s} \frac{\tau_s}{\sigma^2} \frac{e^{-\frac{d_{3q}^2}{2}}}{\sqrt{2\pi}} d_{3q} \\
& + \tau^2 K e^{-r\tau} \Lambda_{2q} + \tau K e^{-r\tau} \frac{1}{\pi\sigma} \int_{-\infty}^{\infty} e^{-\frac{\beta^2}{\tau}} e^{-\frac{d_{6q}^2}{2}} d\beta \\
& + \frac{\tau K e^{-r\tau}}{\pi\sigma} \int_{-\infty}^{\infty} e^{-\frac{\beta^2}{\tau}} e^{-\frac{d_{6q}^2}{2}} d\beta + \frac{K\sqrt{2\tau}}{\pi\sigma^2} e^{-r\tau} \int_{-\infty}^{\infty} e^{-\frac{\beta^2}{\tau}} e^{-\frac{d_{6q}^2}{2}} d_{6q}(\beta) d\beta \\
& - S_0 e^{-q\tau} \frac{\sqrt{2\tau}}{\pi\sigma^2} \int_{-\infty}^{\infty} e^{-\frac{\beta^2}{\tau} - \frac{\sigma^2\tau}{4} + \sigma\beta} e^{-\frac{d_{5q}^2}{2}} d_{5q}(\beta) d\beta. \tag{5.55}
\end{aligned}$$

After collecting similar terms, we get

$$\begin{aligned}
\frac{d^2(S_P)}{dr^2} = & \left( K\tau_s^2\Phi(-d_{4q}) + \frac{2K\sigma\tau_s\sqrt{\tau_s} + K\tau_sd_{4q}}{\sigma^2\sqrt{2\pi}}e^{-\frac{1}{2}d_{4q}^2} \right) e^{-r\tau_s} \\
& - S_0e^{-q\tau_s}\frac{\tau_s}{\sigma^2\sqrt{2\pi}}d_{3q}e^{-\frac{1}{2}(d_{3q})^2} \\
& + Ke^{-r\tau}\left( \tau^2\Lambda_{2q} + \frac{2\tau}{\pi\sigma}\int_{-\infty}^{\infty}e^{-\frac{\beta^2}{\tau}-\frac{1}{2}(d_{6q}(r,\beta))^2}d\beta + \frac{\sqrt{2\tau}}{\pi\sigma^2}\int_{-\infty}^{\infty}e^{-\frac{\beta^2}{\tau}-\frac{1}{2}(d_{6q}(r,\beta))^2}d_{6q}(r,\beta)d\beta \right) \\
& - S_0e^{-q\tau}\frac{\sqrt{2\tau}}{\pi\sigma^2}\int_{-\infty}^{\infty}e^{-\frac{\beta^2}{\tau}-\frac{\sigma^2\tau}{4}+\sigma\beta-\frac{1}{2}(d_{5q}(r,\beta))^2}d_{5q}(r,\beta)d\beta.
\end{aligned} \tag{5.56}$$

We can now approximate the puttable Sukūk price by using its duration and convexity to build a second order Taylor polynomial as follows

$$S_P \approx S_P(r_0) + \frac{d(S_P)}{dr}(r_0)(r - r_0) + \frac{1}{2}\frac{d^2(S_P)}{dr^2}(r_0)(r - r_0)^2. \tag{5.57}$$

Also, the approximation of the puttable Sukūk price by using its modified duration and convexity to build a second order polynomial as follows

$$S_P \approx S_P(r_0) + \frac{1}{(1 + r_0)}\left( \frac{d(S_P)}{dr}(r_0)(r - r_0) + \frac{1}{2}\frac{d^2(S_P)}{dr^2}(r_0)(r - r_0)^2 \right). \tag{5.58}$$

The quadratic terms in Equations (5.57), (5.58) are called the convexity estimate by duration and modified duration respectively. As we can see in Figures 5.9 and 5.10 that adding the convexity estimate gives a better approximation to the puttable Sukūk price, especially when the curvature of the price is not very pronounced. When the puttable Sukūk price has a large curvature, the convexity estimate provides a good approximation only for small intervals around a specific Iājārah rate  $r_0$ . The price and yield of Ijārah are inversely related but not in a linear fashion. This is also clear from Figures 5.9 and 5.10 that if the Ijārah rate of Sukuk fall, then its price will increase and vice versa. These plots give us the sensitivity of puttable Sukūk price to change in Ijārah rate. It must be

mentioned that before mid-term, the puttable Sukūk price is more sensitive to change in Ijārah than after mid-term. This can be justified from Figures 5.9 and 5.10 where one can observe that the Sukūk price-Ijārah curve is more positively convex in Figure 5.9 (that is, at a time before mid-term) than in Figure 5.10 (that is, at a time after mid-term).

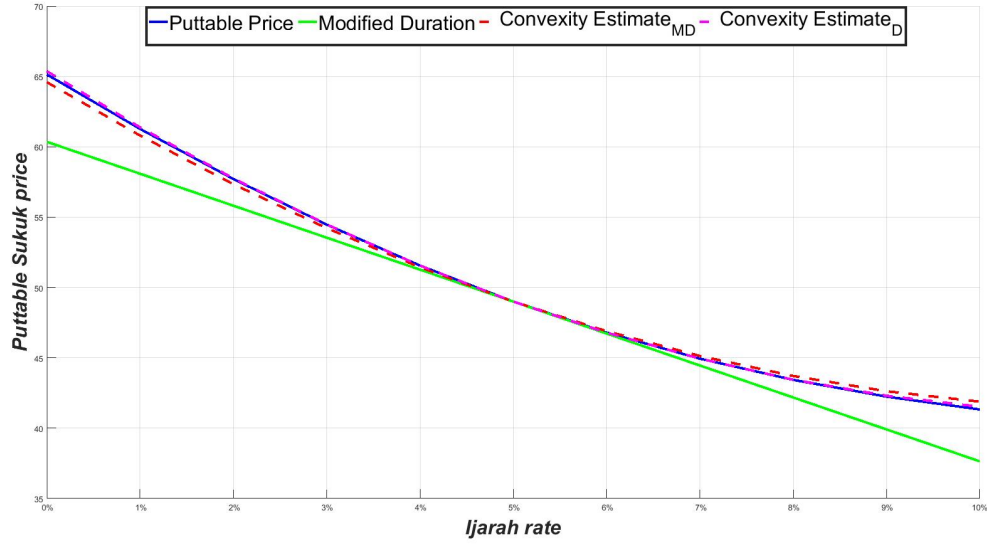


Figure 5.9: Convexity of puttable Sukūk price-Ijārah relationship estimated by duration and modified duration when  $t \leq \frac{T}{2}$

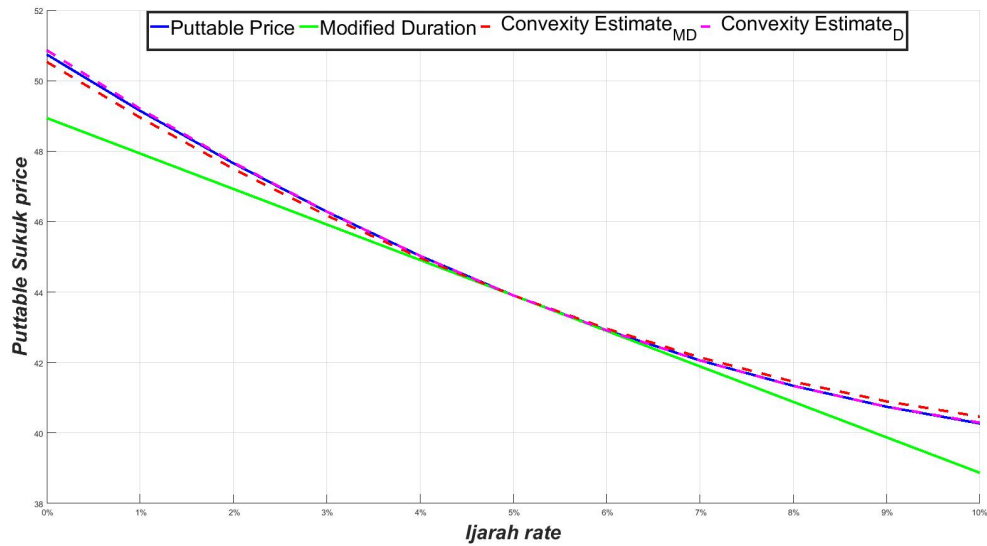


Figure 5.10: Convexity of puttable Sukūk price-Ijārah relationship estimated by duration and modified duration when  $t > \frac{T}{2}$



The positive convexity of the puttable Sukūk price-Ijārah curve is due to the fact that puttable Sukūk is a combination of a straight Sukūk and a put option both of which have a positive convexity. In other words, if the Ijārah rate increases, the chance that the holder of Sukūk will demand the Sukūk increases. Therefore the rate of increase of puttable Sukūk price is more than that for the non-puttable Sukūk as the price decreases. This results in positive convexity in the puttable Sukūk price-Ijārah curve.

Both the puttable Sukūk and the puttable bond have positive convexity. It was mentioned that at a time before mid-term, the puttable Sukūk price-Ijārah curve is more curved than at a time after mid-term. If we compare with the puttable bonds, this is in line with the fact that the puttable bonds with longer maturity are more positively convex than the puttable bonds with less maturity. For a puttable Sukūk as the time to maturity decreases, the Sukūk price-Ijārah curve becomes more flat.

# Chapter 6

## Discussions and Further Work

### 6.1 Discussions

In this thesis we obtained the pricing formula for mid-term call and put option of Ijārah Sukūk for lifespan. These formulae would be useful in practice and help in the actual trading of these options in the Islamic financial markets. In this thesis we not only derived the pricing formulae for mid-term call and put option of Ijārah Sukūk but have also compared them with the European and American options and results are in line with the intuitive results that one would expect. For example, since the mid-term Ijārah Sukūk option gives the holder more right to exercise than the European option but less right than the American option, one would expect that the prices of mid-term call or put options must lie between the prices of European and American call options. We proved that the price of a mid-term call option of Ijārah Sukūk at any time before mid-term lies between the prices of European and American call options. This is in agreement with the intuitive result. After mid-term, the price of the call option of Ijārah Sukūk is the same as that of the European call option. A similar result was obtained for the put option of Ijārah Sukūk.

We further studied the pricing formulae for callable and puttable Ijārah Sukūk. It was again noted that the price of a callable Sukūk lies between that of the European and American callable bonds before mid-term and after mid-term it coincides with that of European callable bond. Also the price of puttable Ijārah Sukūk is less than that of the American puttable bond. We studied duration and convexity to understand the nature of the callable Sukūk price-Ijārah rate curve. It was observed that the callable Sukūk price-Ijārah rate curve is negatively convex while the puttable Sukūk price- Ijārah rate curve is positively convex. This is in agreement with associated results on callable and puttable Bonds.

The study of callable and puttable Sukūk will be helpful in actual trading of these instruments and generating liquidity in the markets. This is only a small step in popularisation of Sukūk options and which will benefit the Islamic financial world. It helps the Muslim population to participate in the economic and financial world and benefit in the same way as any other investor but still following the principles of Islamic Shariah law.

## 6.2 Further Work

The results obtained in this thesis can further be improved as well as some new research directions can be pursued from the questions left unanswered. Throughout the thesis we assumed that the underlying follows Geometric Brownian motion. This assumption is very important for obtaining nice pricing formulae as in the Black-Scholes theory but for practical purposes are not very useful. For example, the assumptions of the Black-Scholes model can be relaxed to get better results. The mid-term Sukūk options pricing formulae obtained are based on the hypothesis that the volatility of the underlying asset is constant. In the future work, we propose to adopt stochastic volatility models, like Heston Model (1993), to price Sukūk options. Hence, I will derive a model

based on the Heston model and compare it with the Sukūk price options equation, and make a sensitivity analysis for its parameters.

In callable and puttable Sukūk price model, I assumed that the straight Sukūk price is the face value. But since the Sukūk holders share the risk of the underlying asset, the initial investment is not guaranteed. This means in general Sukūk holder may or may not get back the entire principal (face value) amount. Therefore, the assumption that the straight Sukūk price is the face value is a limitation of my research. So, in my future work I would assume the straight Sukūk price is based on the market price at that time.

Furthermore, I will do the same options model for other types of Sukūk like for Musharakah Sukūk. However, in Musharakah Sukūk, the return to the investors is not fixed in the beginning of contract as in Ijārah Sukūk but the return is based on the profit/loss in the project that the issuer and investor partner in it. Hence, the return to the investor will be stochastic. It means that the model will be stochastic model.

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